Bisimilar finite abstractions of stochastic control systems

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***Abstract*— Abstraction-based approaches to the design of complex control systems construct finite-state models that are formally related to the original control systems, then leverage symbolic techniques from finite-state synthesis to compute controllers satisfying specifications given in a temporal logic, and finally refine the obtained control schemes back to the given concrete complex models. While such approaches have been successfully used to perform synthesis over non-probabilistic control systems, there are only few results available for probabilistic models: hence the goal of this paper, which considers continuous-time controlled stochastic differential equations. We show that for every stochastic control system satisfying a stochastic version of incremental input-to-state stability, and for every $\varepsilon > 0$, there exists a finite-state abstraction that is ε-approximate bisimilar to the stochastic control system (in the sense of moments). We demonstrate the effectiveness of the construction by synthesizing a controller for a stochastic control system with respect to linear temporal logic specifications. Since stochastic control systems are a common mathematical models for many complex, safety critical systems subject to uncertainty, our techniques promise to enable a new, automated, correct-by-construction controller synthesis approach for these systems.**

I. Introduction

The synthesis of controllers over complex hybrid control systems with respect to rich specifications is a grand challenge in cyber-physical systems research. One recent approach in this direction is the use of *symbolic models*, which represent discrete and finite approximations of more complex (e.g., continuous and uncountable) model dynamics, and which allow relating a controller designed for the symbolic approximation to one for the original dynamics. Whenever finite symbolic models of complex dynamics can be constructed, one can use synthesis techniques for finite-state models [18] to design controllers for the concrete models. The formal notion of approximation is cast via ε-approximate bisimulation relations [9], which guarantee that each trace of the continuous system can be matched by a trace of the symbolic model up to a precision ε, and vice versa.

The construction of finite symbolic models has been widely investigated over non-probabilistic control systems [7], [10], [11], [17], [22], [23], [24], [26], [29], [31]. For example, ε-approximate bisimilar symbolic models exist for non-probabilistic incremental input-to-state stable control systems, for any precision $\varepsilon > 0$ [10], [17], [22]. Symbolic models form the basis of controller synthesis tools such as Pessoa [19].

On the other hand, the important class of continuous *stochastic* control systems have not yet been looked at from the perspective of finite symbolic abstractions. Existing results on stochastic systems include the construction of finite bisimilar abstractions for continuous-time diffusions under some contractivity assumption [1], for discrete-time stochastic hybrid dynamical systems endowed with certain continuity and ergodic properties [3], and for discrete-time stochastic dynamical systems under a notion of bisimulation function [6]. All these techniques are restricted to *autonomous* models (that is with no control inputs), and as such they can be used for verification, but not for controller synthesis.

For non-autonomous models there are techniques to check if a lower dimensional abstraction (still uncountable) is formally related to a concrete stochastic control model by the notion of approximate probabilistic bisimulation [13], however these results do not extend to the *construction* of approximations, nor do they deal with *finite* abstractions, and finally appear to be computationally tractable only when restricted to the autonomous case. For specific properties expressed in the PCTL logic, there are techniques to compute finite abstractions of discrete-time stochastic control models [2], [4], but their generalization to linear temporal logic is not obvious. The work in [27] provides schemes for control problems over probabilistic rectangular automata, in which random behaviors occur only over the discrete components. The work in [16] presents a finite MDP abstraction of a continuous-time controlled diffusion for the verification of given temporal properties, however the relationship between abstract and concrete model is not quantitative – similarly, classical discretization results in the literature [15] lead to discrete approximations of continuous models with asymptotic convergence, rather than with formal relationships based on bisimulation or simulation notions that can ensure the correspondence of controllers on linear temporal logic specifications over model trajectories. In conclusion, there seems to be no comprehensive work on the construction of finite bisimilar abstractions for continuous-time stochastic control systems, which represent models for cyber-physical systems operating in an uncertain or noisy environment, and for which automated synthesis methodologies can allow for a more effective model development and regulation.

In this paper, we show the existence of ε-approximate bisimilar symbolic models for continuous-time stochastic

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control systems satisfying a stochastic version of the incremental input-to-state stability property [5], for any parameter $\varepsilon > 0$. Furthermore, we show that the symbolic models are finite if the continuous states lie within a bounded set. We also provide a simple way to construct the symbolic abstractions by quantizing the state and input sets. Since we guarantee ε-approximate bisimulation, there exists a controller enforcing a desired specification on the symbolic model if and only if there exists a controller enforcing an ε-related specification on the original stochastic control system. Our construction nicely generalizes the construction for non-probabilistic systems [10], [17], [22], tailoring to the results for non-probabilistic systems in the special case with no noise. We illustrate our results on an example, where a controller is synthesized for a stochastic control system with respect to specifications expressed in linear temporal logic.

II. Stochastic Control Systems

A. Notations

The identity map on a set A is denoted by 1_A. If A is a subset of B we denote by $\imath_A : A \hookrightarrow B$ or simply by $\imath$ the natural inclusion map taking any $a \in A$ to $\imath(a) = a \in B$. The symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ denote the set of natural, nonnegative integer, integer, real, positive, and nonnegative real numbers, respectively. The symbols I_n, 0_n, and $0_{n\times m}$ denote the identity matrix, the zero vector and zero matrix in $\mathbb{R}^{n\times n}$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times m}$, respectively. Given a vector $x \in \mathbb{R}^n$, we denote by x_i the i–th element of x, and by $\|x\|$ the infinity norm of x, namely $\|x\| = \max\{|x_1|, |x_2|, ..., |x_n|\}$, where $|x_i|$ denotes the absolute value of x_i. Given a matrix $M = \{m_{ij}\} \in \mathbb{R}^{n\times m}$, we denote by $\|M\|$ the infinity norm of M, namely, $\|M\| = \max_{1\leq i\leq n}\sum_{j=1}^{m}|m_{ij}|$, and by $\|M\|_F$ the Frobenius norm of M, namely, $\|M\|_F = \sqrt{\text{Tr}\,(MM^T)}$, where $\text{Tr}(P) = \sum_{i=1}^{n} p_{ii}$ for any $P = \{p_{ij}\} \in \mathbb{R}^{n\times n}$. We denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ the minimum and maximum eigenvalues of matrix A, respectively. The diagonal set $\Delta \subset \mathbb{R}^{2n}$ is defined as: $\Delta = \{(x,x) \mid x \in \mathbb{R}^n\}$.

The closed ball centered at $x \in \mathbb{R}^n$ with radius ε is defined by $\mathcal{B}_\varepsilon(x) = \{y \in \mathbb{R}^n \mid \|x-y\| \leq \varepsilon\}$. A set $B \subseteq \mathbb{R}^n$ is called a *box* if $B = \prod_{i=1}^{n}[c_i, d_i]$, where $c_i, d_i \in \mathbb{R}$ with $c_i < d_i$ for each $i \in \{1, \ldots, n\}$. The *span* of a box B is defined as $\mathit{span}(B) = \min\{|d_i - c_i| \mid i = 1, \ldots, n\}$. Define the η-approximation $[B]_\eta = \{b \in B \mid b_i = k_i\eta,\ k_i \in \mathbb{Z},\ i = 1, \ldots, n\}$ for a box B and $\eta \leq \mathit{span}(B)$. Note that $[B]_\eta \neq \varnothing$ for any $\eta \leq \mathit{span}(B)$. Geometrically, for any $\eta \in \mathbb{R}^+$ with $\eta \leq \mathit{span}(B)$ and $\lambda \geq \eta$, the collection of sets $\{\mathcal{B}_\lambda(p)\}_{p\in[B]_\eta}$ is a finite covering of B, i.e., $B \subseteq \bigcup_{p\in[B]_\eta}\mathcal{B}_\lambda(p)$. By defining $[\mathbb{R}^n]_\eta = \{a \in \mathbb{R}^n \mid a_i = k_i\eta,\ k_i \in \mathbb{Z},\ i = 1, \cdots, n\}$, the set $\bigcup_{p\in[\mathbb{R}^n]_\eta}\mathcal{B}_\lambda(p)$ is a countable covering of $\mathbb{R}^n$ for any $\eta \in \mathbb{R}^+$ and $\lambda \geq \eta$. We extend the notions of *span* and approximation to finite unions of boxes as follows. Let $A = \bigcup_{j=1}^{M} A_j$, where each A_j is a box. Define $\mathit{span}(A) = \min\{\mathit{span}(A_j) \mid j = 1, \ldots, M\}$, and for any $\eta \leq \mathit{span}(A)$, define $[A]_\eta = \bigcup_{j=1}^{M}[A_j]_\eta$.

Given a set X, a function $\mathbf{d} : X \times X \rightarrow \mathbb{R}_0^+$ is a metric on X if for any $x, y, z \in X$, the following three conditions are satisfied: i) $\mathbf{d}(x,y) = 0$ if and only if $x = y$; ii) $\mathbf{d}(x,y) = \mathbf{d}(y,x)$; and iii) (triangle inequality) $\mathbf{d}(x,z) \leq \mathbf{d}(x,y) + \mathbf{d}(y,z)$. Given a measurable function $f : \mathbb{R}_0^+ \rightarrow \mathbb{R}^n$, the (essential) supremum of f is denoted by $\|f\|_\infty$; we recall that $\|f\|_\infty := \text{(ess)sup}\{\|f(t)\|, t \geq 0\}$. A function f is essentially bounded if $\|f\|_\infty < \infty$. For a given time $\tau \in \mathbb{R}^+$, define f_τ so that $f_\tau(t) = f(t)$, for any $t \in [0, \tau)$, and $f_\tau(t) = 0$ elsewhere; f is said to be locally essentially bounded if for any $\tau \in \mathbb{R}^+$, f_τ is essentially bounded. A continuous function $\gamma : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \rightarrow \infty$ as $r \rightarrow \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the map $\beta(r,s)$ belongs to class $\mathcal{K}_\infty$ with respect to r and, for each fixed nonzero r, the map $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s) \rightarrow 0$ as $s \rightarrow \infty$. We identify a relation $R \subseteq A \times B$ with the map $R : A \rightarrow 2^B$ defined by $b \in R(a)$ iff $(a,b) \in R$. Given a relation $R \subseteq A \times B$, R^{-1} denotes the inverse relation defined by $R^{-1} = \{(b,a) \in B \times A : (a,b) \in R\}$.

B. Stochastic control systems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_s)_{s\geq 0}$ satisfying the usual conditions of completeness and right continuity [14, p. 48]. Let $(W_s)_{s\geq 0}$ be a p-dimensional $\mathbb{F}$-Brownian motion.

Definition 2.1: A *stochastic control system* is a tuple $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f, \sigma)$, where

- $\mathbb{R}^n$ is the state space;
- $\mathsf{U} \subseteq \mathbb{R}^m$ is an input set;
- $\mathcal{U}$ is a subset of the set of all measurable, locally essentially bounded functions of time from intervals of the form $[0, \infty[$ to U;
- $f : \mathbb{R}^n \times \mathsf{U} \rightarrow \mathbb{R}^n$ is a continuous function of its arguments satisfying the following Lipschitz assumption: there exist constants $L_x, L_u \in \mathbb{R}^+$ such that: $\|f(x,u) - f(x',u')\| \leq L_x\|x - x'\| + L_u\|u - u'\|$ for all $x, x' \in \mathbb{R}^n$ and all $u, u' \in \mathsf{U}$;
- $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^{n\times p}$ is a continuous function satisfying the following Lipschitz assumption: there exists a constant $Z \in \mathbb{R}^+$ such that: $\|\sigma(x) - \sigma(x')\| \leq Z\|x - x'\|$ for all $x, x' \in \mathbb{R}^n$. □

A stochastic process $\xi : \Omega \times [0, \infty[\rightarrow \mathbb{R}^n$ is said to be a *solution process* of Σ if there exists $\upsilon \in \mathcal{U}$ satisfying:

$$\mathrm{d}\,\xi = f(\xi, \upsilon)\,\mathrm{d}\,t + \sigma(\xi)\,\mathrm{d}\,W_t, \qquad \text{(II.1)}$$

$\mathbb{P}$-almost surely ($\mathbb{P}$-a.s.), where f is known as the drift, σ as the diffusion, and again W_t is Brownian motion. We also write $\xi_{a\upsilon}(t)$ to denote the value of the solution process at time $t \in \mathbb{R}_0^+$ under the input curve υ from initial condition $\xi_{a\upsilon}(0) = a$ $\mathbb{P}$-a.s., in which a is a random variable that is measurable in $\mathcal{F}_0$. Let us remark that $\mathcal{F}_0$ in general is not a trivial sigma-algebra, and thus the stochastic control system Σ can start from a random initial condition. Let us emphasize that the solution process is uniquely determined, since the assumptions on f and σ ensure its existence and uniqueness [21, Theorem 5.2.1, p. 68].

III. A Notion of Incremental Stability

This section introduces a stability notion for stochastic control systems that generalizes the notion of incremental input-to-state stability (δ-ISS) [5] for non-probabilistic control systems.

Definition 3.1: A stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f, \sigma)$ is incrementally input-to-state stable in the qth moment (δ-ISS-M$_q$), where $q \geq 1$, if there exist a $\mathcal{KL}$ function β and a $\mathcal{K}_\infty$ function γ such that for any $t \in \mathbb{R}_0^+$, any $\mathbb{R}^n$-valued random variables a and a' that are measurable in $\mathcal{F}_0$, and any $\upsilon, \upsilon' \in \mathcal{U}$, the following condition is satisfied:

$$\mathbb{E}\left[\|\xi_{a\upsilon}(t) - \xi_{a'\upsilon'}(t)\|^q\right] \leq \beta\left(\mathbb{E}\left[\|a - a'\|^q\right], t\right) + \gamma\left(\|\upsilon - \upsilon'\|_\infty\right). \quad \text{(III.1)}$$

□

It can be easily checked that a δ-ISS-M$_q$ stochastic control system Σ is δ-ISS in the absence of any noise as in the following:

$$\|\xi_{a\upsilon}(t) - \xi_{a'\upsilon'}(t)\| \leq \beta\left(\|a - a'\|, t\right) + \gamma\left(\|\upsilon - \upsilon'\|_\infty\right), \quad \text{(III.2)}$$

for $a, a' \in \mathbb{R}^n$, some $\beta \in \mathcal{KL}$, and some $\gamma \in \mathcal{K}_\infty$. Moreover, whenever $f(0_n, 0_m) = 0_n$ and $\sigma(0_n) = 0_{n\times p}$ (i.e., the drift and diffusion terms vanish at the origin), then δ-ISS-M$_q$ implies input-to-state stability in the qth moment (ISS-M$_q$) [12] and global asymptotic stability in the qth moment (GAS-M$_q$) [8].

We describe the notion of δ-ISS-M$_q$ in terms of the existence of *incremental Lyapunov functions*, along the same line as for δ-ISS Lyapunov functions in the non-probabilistic case [5].

Definition 3.2: Consider a stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f, \sigma)$ and a continuous function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ that is smooth on $\{\mathbb{R}^n \times \mathbb{R}^n\}\backslash\Delta$. The function V is called an incremental input-to-state stable in the qth moment (δ-ISS-M$_q$) Lyapunov function for Σ, where $q \geq 1$, if there exist $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, ρ, and a constant $\kappa \in \mathbb{R}^+$, such that

(i) $\underline{\alpha}$ (resp. $\overline{\alpha}$) is a convex (resp. concave) function;
(ii) for any $x, x' \in \mathbb{R}^n$, $\underline{\alpha}\left(\|x - x'\|^q\right) \leq V(x, x') \leq \overline{\alpha}\left(\|x - x'\|^q\right)$;
(iii) for any $x, x' \in \mathbb{R}^n$, $x \neq x'$, and for any $u, u' \in \mathsf{U}$,

$$\begin{aligned}\mathcal{L}^{u,u'}V(x,x') &:= \begin{bmatrix}\partial_x V & \partial_{x'} V\end{bmatrix}\begin{bmatrix} f(x,u) \\ f(x',u')\end{bmatrix} \\ &+ \frac{1}{2}\text{Tr}\left(\begin{bmatrix}\sigma(x) \\ \sigma(x')\end{bmatrix}\begin{bmatrix}\sigma^T(x) & \sigma^T(x')\end{bmatrix}\begin{bmatrix}\partial_{x,x} V & \partial_{x,x'} V \\ \partial_{x',x} V & \partial_{x',x'} V\end{bmatrix}\right) \\ &\leq -\kappa V(x,x') + \rho(\|u - u'\|),\end{aligned}$$

where $\mathcal{L}^{u,u'}$ is the infinitesimal generator associated to the stochastic control system (II.1) [21, Section 7.3], which in this case depends on two separate controls u, u'. The symbols ∂_x and $\partial_{x,x'}$ denote first- and second-order partial derivatives with respect to x and x', respectively. □

In words, condition (ii) implies that the growth rate of functions $\overline{\alpha}$ and $\underline{\alpha}$ are linear, as a concave function is supposed to dominate a convex one. However, these conditions do not restrict the behavior of $\overline{\alpha}$ and $\underline{\alpha}$ to only linear functions on a compact subset of $\mathbb{R}^n$. The following theorem describes δ-ISS-M$_q$ in terms of the existence of δ-ISS-M$_q$ Lyapunov functions.

Theorem 3.3: A stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f, \sigma)$ is δ-ISS-M$_q$ if it admits a δ-ISS-M$_q$ Lyapunov function. □

The proof of Theorem 3.3 is provided in [30]. One can resort to available software tools, such as SOSTOOLS [25], to search for appropriate, non-trivial δ-ISS-M$_q$ Lyapunov functions for system Σ. Alternatively, we look into special instances where function V can be easily computed based on the model dynamics. The first result provides a sufficient condition for a particular function V to be a δ-ISS-M$_q$ Lyapunov function for a stochastic control system Σ, when $q = 1, 2$ (that is, in the first or second moment).

Lemma 3.4: Consider a stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f, \sigma)$. Let $q \in \{1, 2\}$, $P \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix, and the function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ be defined as follows:

$$V(x, x') := \left(\widetilde{V}(x, x')\right)^{\frac{q}{2}} = \left(\frac{1}{q}\left(x - x'\right)^T P\left(x - x'\right)\right)^{\frac{q}{2}}, \quad \text{(III.3)}$$

and satisfy

$$(x - x')^T P(f(x,u) - f(x',u)) + \frac{1}{2}\left\|\sqrt{P}\left(\sigma(x) - \sigma(x')\right)\right\|_F^2 \leq -\widetilde{\kappa}\left(V(x,x')\right)^{\frac{2}{q}}, \quad \text{(III.4)}$$

or, if f is differentiable with respect to x, satisfy

$$(x - x')^T P\partial_x f(z,u)(x - x') + \frac{1}{2}\left\|\sqrt{P}\left(\sigma(x) - \sigma(x')\right)\right\|_F^2 \leq -\widetilde{\kappa}\left(V(x,x')\right)^{\frac{2}{q}}, \quad \text{(III.5)}$$

for all $x, x', z \in \mathbb{R}^n$, for all $u \in \mathsf{U}$, and for some constant $\widetilde{\kappa} \in \mathbb{R}^+$. Then V is a δ-ISS-M$_q$ Lyapunov function for Σ. □

The proof of Lemma 3.4 is provided in [30]. The next result provides a condition that is equivalent to (III.4) or (III.5) for linear stochastic control systems Σ (that is, for systems with linear drift and diffusion terms) in the form of a linear matrix inequality (LMI), which can be easily solved numerically.

Corollary 3.5: Consider a stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f, \sigma)$, where for all $x \in \mathbb{R}^n$, and $u \in \mathsf{U}$, $f(x,u) := Ax + Bu$, for some $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $\sigma(x) := [\sigma_1 x \; \sigma_2 x \; \cdots \; \sigma_p x]$, for some $\sigma_i \in \mathbb{R}^{n\times n}$. Then, function V in (III.3) is a δ-ISS-M$_q$ Lyapunov function for Σ, when $q \in \{1, 2\}$, if there exists a constant $\widehat{\kappa} \in \mathbb{R}^+$ satisfying the following LMI:

$$PA + A^T P + \sum_{i=1}^{p} \sigma_i^T P \sigma_i \preceq -\widehat{\kappa} P. \quad \text{(III.6)}$$

□

The proof of Corollary 3.5 is provided in [30]. As a practical consequence of the previous corollary, in order to obtain tighter upper bounds in (III.1) one can seek appropriate matrices P by solving appropriately the LMI in (III.6).

A. Noisy and noise-free trajectories

In order to introduce a symbolic model in Section V for the stochastic control system, we need the following technical results, which provide an upper bound on the distance (in the qth moment) between the solution processes of Σ and those of the corresponding non-probabilistic control system obtained by disregarding the diffusion term (that is, σ).

Lemma 3.6: Consider a stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f, \sigma)$ such that $f(0_n, 0_m) = 0_n$, and $\sigma(0_n) = 0_{n\times p}$. Suppose there exists a δ-ISS-M$_q$ Lyapunov function V for Σ such that its Hessian is a positive semidefinite matrix in $\mathbb{R}^{2n\times 2n}$ and $q \geq 2$. Then for any x in a compact set D and any $\upsilon \in \mathcal{U}$, we have

$$\mathbb{E}\left[\left\|\xi_{x\upsilon}(t) - \overline{\xi}_{x\upsilon}(t)\right\|^q\right] \leq h(\sigma, t), \tag{III.7}$$

where $\overline{\xi}_{x\upsilon}$ is the solution of the ordinary differential equation (ODE) $\dot{\overline{\xi}}_{x\upsilon}(t) = f\left(\overline{\xi}_{x\upsilon}(t), \upsilon(t)\right)$ starting from the initial condition x, and the nonnegative valued function h tends to zero as $t \to 0$, $t \to +\infty$, or as $Z \to 0$, where Z is the Lipschitz constant, introduced in Definition 2.1. □

The proof of Lemma 3.6 is provided in [30]. In particular, one can compute explicitly the function h using equation (IX.4) in [30]. Moreover, we refer the interested readers to Lemma 3.9 and Corollary 3.10 in [30], providing explicit forms of the function h for (linear) stochastic control systems Σ admitting a δ-ISS-M$_q$ Lyapunov function V as in (III.3), where $q \in \{1, 2\}$.

IV. Symbolic Models

A. Systems

We employ the notion of system [28] to describe both stochastic control systems as well as their symbolic models.

Definition 4.1: A system S is a tuple $S = (X, X_0, U, \longrightarrow, Y, H)$, where X is a set of states, $X_0 \subseteq X$ is a set of initial states, U is a set of inputs, $\longrightarrow \subseteq X \times U \times X$ is a transition relation, Y is a set of outputs, and $H : X \to Y$ is an output map. □

A transition $(x, u, x') \in \longrightarrow$ is also denoted by $x \xrightarrow{u} x'$. For a transition $x \xrightarrow{u} x'$, state x' is called a u-successor, or simply a successor, of state x. For technical reasons, we assume that for any $x \in X$, there exists some u-successor of x for some $u \in U$ – let us remark that this is always the case for the considered systems later in this paper.

System S is said to be

- *metric*, if the output set Y is equipped with a metric $\mathbf{d} : Y \times Y \to \mathbb{R}_0^+$;
- *finite*, if X is a finite set;
- *deterministic*, if for any state $x \in X$ and any input u, there exists exactly one u-successor.

For a system $S = (X, X_0, U, \longrightarrow, Y, H)$ and given any state $x_0 \in X_0$, a finite state run generated from x_0 is a finite sequence of transitions: $x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} x_2 \xrightarrow{u_2} \cdots \xrightarrow{u_{n-2}} x_{n-1} \xrightarrow{u_{n-1}} x_n$, such that $x_i \xrightarrow{u_i} x_{i+1}$ for all $0 \leq i < n$. A finite state run can be directly extended to an infinite state run as well.

B. System relations

We recall the notion of approximate (bi)simulation relation, introduced in [9], which is useful when analyzing or synthesizing controllers for deterministic systems.

Definition 4.2: Let $S_a = (X_a, X_{a0}, U_a, \underset{a}{\longrightarrow}, Y_a, H_a)$ and $S_b = (X_b, X_{b0}, U_b, \underset{b}{\longrightarrow}, Y_b, H_b)$ be metric systems with the same output sets $Y_a = Y_b$ and metric $\mathbf{d}$. For $\varepsilon \in \mathbb{R}^+$, a relation $R \subseteq X_a \times X_b$ is said to be an ε-approximate simulation relation from S_a to S_b if the following three conditions are satisfied:

(i) for every $x_{a0} \in X_{a0}$, there exists $x_{b0} \in X_{b0}$ with $(x_{a0}, x_{b0}) \in R$;
(ii) for every $(x_a, x_b) \in R$ we have $\mathbf{d}(H_a(x_a), H_b(x_b)) \leq \varepsilon$;
(iii) for every $(x_a, x_b) \in R$ we have that $x_a \underset{a}{\xrightarrow{u_a}} x'_a$ in S_a implies the existence of $x_b \underset{b}{\xrightarrow{u_b}} x'_b$ in S_b satisfying $(x'_a, x'_b) \in R$.

A relation $R \subseteq X_a \times X_b$ is said to be an ε-approximate bisimulation relation between S_a and S_b if R is an ε-approximate simulation relation from S_a to S_b and R^{-1} is an ε-approximate simulation relation from S_b to S_a.

System S_a is ε-approximately simulated by S_b, or S_b ε-approximately simulates S_a, denoted by $S_a \preceq_{\mathcal{S}}^{\varepsilon} S_b$, if there exists an ε-approximate simulation relation from S_a to S_b. System S_a is ε-approximate bisimilar to S_b, denoted by $S_a \cong_{\mathcal{S}}^{\varepsilon} S_b$, if there exists an ε-approximate bisimulation relation R between S_a and S_b. □

Note that when $\varepsilon = 0$, condition (ii) in the above definition is modified as $(x_a, x_b) \in R$ if and only if $H_a(x_a) = H_b(x_b)$, and R becomes an exact simulation relation, as introduced in [20]. Similarly, whenever $\varepsilon = 0$, R possibly becomes an exact bisimulation relation.

V. Symbolic Models for Stochastic Control Systems

This section contains the main results of the paper, showing that for any stochastic control system Σ admitting a δ-ISS-M$_q$ Lyapunov function, and for any precision level $\varepsilon \in \mathbb{R}^+$, we can construct a finite system that is ε-approximately bisimilar to Σ. In order to do so, we use the notion of system as an abstract representation of a stochastic control system, capturing all the information contained in it. More precisely, given a stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f, \sigma)$, we define an associated metric system $S(\Sigma) = (X, X_0, U, \longrightarrow, Y, H)$, where:

- X is the set of all $\mathbb{R}^n$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- X_0 is the set of all $\mathbb{R}^n$-valued random variables that are measurable over the trivial sigma-algebra $\mathcal{F}_0$, i.e., the system starts from a deterministic initial condition,

which is equivalently a random variable with a Dirac probability distribution;
- $U = \mathcal{U}$;
- $x \xrightarrow{\upsilon} x'$ if x and x' are measurable in $\mathcal{F}_t$ and $\mathcal{F}_{t+\tau}$, respectively, for some $t \in \mathbb{R}_0^+$ and $\tau \in \mathbb{R}^+$, and there exists a solution process $\xi : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^n$ of Σ satisfying $\xi(t) = x$ and $\xi_{x\upsilon}(\tau) = x'$ $\mathbb{P}$-a.s.;
- Y is the set of all $\mathbb{R}^n$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- $H = 1_X$.

We assume that the output set Y is equipped with the natural metric $\mathbf{d}(y, y') = \left(\mathbb{E}\left[\|y - y'\|^q\right]\right)^{\frac{1}{q}}$, for any $y, y' \in Y$ and some $q \geq 1$. Let us remark that the set of states of $S(\Sigma)$ is uncountable and that $S(\Sigma)$ is a deterministic system in the sense of Definition 4.1, since (cf. Subsection II-B) the solution process of Σ is uniquely determined.

The results in this section rely on additional assumptions on model Σ that are described next (however such assumptions are not required for the definitions and results in Sections II, III, and IV). We restrict our attention to stochastic control systems $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f, \sigma)$ with $f(0_n, 0_m) = 0_n$, $\sigma(0_n) = 0_{n\times p}$, and input sets U that are assumed to be finite unions of boxes. We further restrict our attention to sampled-data stochastic control systems, where input curves belong to set $\mathcal{U}_\tau$ which contains only curves that are constant over intervals of length $\tau \in \mathbb{R}^+$, i.e.

$$\mathcal{U}_\tau = \Big\{\upsilon : \mathbb{R}_0^+ \to \mathsf{U} \mid \upsilon(t) = \upsilon((k-1)\tau), \\ t \in [(k-1)\tau, k\tau[, k \in \mathbb{N}\Big\}.$$

Let us denote by $S_\tau(\Sigma)$ a sub-system of $S(\Sigma)$ obtained by selecting those transitions from $S(\Sigma)$ corresponding to solution processes of duration τ and to control inputs in $\mathcal{U}_\tau$. This can be seen as the time discretization or as the sampling of a process. More precisely, given a stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f, \sigma)$, we define the associated metric system $S_\tau(\Sigma) = \left(X_\tau, X_{\tau 0}, U_\tau, \underset{\tau}{\longrightarrow}, Y_\tau, H_\tau\right)$, where $X_\tau = X$, $X_{\tau 0} = X_0$, $U_\tau = \mathcal{U}_\tau$, $Y_\tau = Y$, $H_\tau = H$, and

- $x_\tau \xrightarrow[\tau]{\upsilon_\tau} x'_\tau$ if x_τ and x'_τ are measurable, respectively, in $\mathcal{F}_{k\tau}$ and $\mathcal{F}_{(k+1)\tau}$ for some $k \in \mathbb{N}_0$, and there exists a solution process $\xi : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^n$ of Σ satisfying $\xi(k\tau) = x_\tau$ and $\xi_{x_\tau \upsilon_\tau}(\tau) = x'_\tau$ $\mathbb{P}$-a.s..

Notice that a finite state run $x_0 \xrightarrow[\tau]{\upsilon_0} x_1 \xrightarrow[\tau]{\upsilon_1} \ldots \xrightarrow[\tau]{\upsilon_{N-1}} x_N$, of $S_\tau(\Sigma)$, where $\upsilon_i \in \mathcal{U}_\tau$ and $x_i = \xi_{x_{i-1}\upsilon_{i-1}}(\tau)$ for $i = 1, \cdots, N$, captures the solution process of the stochastic control system Σ at times $t = 0, \tau, \cdots, N\tau$, started from the deterministic initial condition x_0 and resulting from a control input υ obtained by the concatenation of the input curves υ_i (i.e. $\upsilon(t) = \upsilon_{i-1}(t)$ for any $t \in [(i-1)\tau, i\,\tau[$), for $i = 1, \cdots, N$.

Let us proceed introducing a fully symbolic system for the concrete model Σ. Consider a stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f, \sigma)$ and a triple $\mathsf{q} = (\tau, \eta, \mu)$ of quantization parameters, where τ is the sampling time, η is the state space quantization, and μ is the input set quantization. Given Σ and q, consider the following system:

$$S_\mathsf{q}(\Sigma) = (X_\mathsf{q}, X_{\mathsf{q}0}, U_\mathsf{q}, \underset{\mathsf{q}}{\longrightarrow}, Y_\mathsf{q}, H_\mathsf{q}), \tag{V.1}$$

consisting of (cf. Notation in Subsection II-A):
- $X_\mathsf{q} = [\mathbb{R}^n]_\eta$;
- $X_{\mathsf{q}0} = [\mathbb{R}^n]_\eta$;
- $U_\mathsf{q} = [\mathsf{U}]_\mu$;
- $x_\mathsf{q} \xrightarrow[\mathsf{q}]{u_\mathsf{q}} x'_\mathsf{q}$ if there exists a $x'_\mathsf{q} \in X_\mathsf{q}$ such that $\left\|\overline{\xi}_{x_\mathsf{q} u_\mathsf{q}}(\tau) - x'_\mathsf{q}\right\| \leq \eta$, where $\dot{\overline{\xi}}_{x_\mathsf{q} u_\mathsf{q}}(t) = f\left(\overline{\xi}_{x_\mathsf{q} u_\mathsf{q}}(t), u_\mathsf{q}(t)\right)$;
- Y_q is the set of all $\mathbb{R}^n$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- $H_\mathsf{q} = \imath : X_\mathsf{q} \hookrightarrow Y_\mathsf{q}$.

Note that we have abused notation by identifying $u_\mathsf{q} \in [\mathsf{U}]_\mu$ with the constant input curve with domain $[0, \ \tau[$ and value u_q. Notice that the proposed abstraction $S_\mathsf{q}(\Sigma)$ is a deterministic system in the sense of Definition 4.1. In order to establish an approximate bisimulation relation, the output set Y_q is defined similarly to the stochastic system $S_\tau(\Sigma)$. Therefore, in the definition of H_q, the inclusion map $\imath$ is meant, with a slight abuse of notation, as a mapping from a deterministic grid point to a random variable with a Dirac probability distribution centered at the grid point. As argued in [28], there is no loss of generality to alternatively assume that $Y_\mathsf{q} = X_\mathsf{q}$ and $H_\mathsf{q} = 1_{X_\mathsf{q}}$.

The transition relation of $S_\mathsf{q}(\Sigma)$ is well defined in the sense that for every $x_\mathsf{q} \in [\mathbb{R}^n]_\eta$ and every $u_\mathsf{q} \in [\mathsf{U}]_\mu$ there always exists $x'_\mathsf{q} \in [\mathbb{R}^n]_\eta$ such that $x_\mathsf{q} \xrightarrow[\mathsf{q}]{u_\mathsf{q}} x'_\mathsf{q}$. This can be seen since by definition of $[\mathbb{R}^n]_\eta$, for any $\widehat{x} \in \mathbb{R}^n$ there always exists a state $\widehat{x}' \in [\mathbb{R}^n]_\eta$ such that $\|\widehat{x} - \widehat{x}'\| \leq \eta$. Hence, for $\overline{\xi}_{x_\mathsf{q} u_\mathsf{q}}(\tau)$ there always exists a state $x'_\mathsf{q} \in [\mathbb{R}^n]_\eta$ satisfying $\left\|\overline{\xi}_{x_\mathsf{q} u_\mathsf{q}}(\tau) - x'_\mathsf{q}\right\| \leq \eta$.

In order to show the first main result of this work, we raise a supplementary assumption on the δ-ISS-M$_q$ Lyapunov function V as follows:

$$|V(x, y) - V(x, z)| \leq \widehat{\gamma}(\|y - z\|), \tag{V.2}$$

for any $x, y, z \in \mathbb{R}^n$, and some $\mathcal{K}_\infty$ and concave function $\widehat{\gamma}$. This assumption is not restrictive, provided V is restricted to a compact subset of $\mathbb{R}^n \times \mathbb{R}^n$. Indeed, for all $x, y, z \in D$, where $D \subset \mathbb{R}^n$ is compact, by applying the mean value theorem to the function $y \to V(x, y)$, one gets

$$|V(x, y) - V(x, z)| \leq \widehat{\gamma}\left(\|y - z\|\right), \\ \text{where} \quad \widehat{\gamma}(r) = \left(\max_{x, y \in D \setminus \Delta} \left\|\frac{\partial V(x, y)}{\partial y}\right\|\right) r.$$

In particular, for the δ-ISS-M$_1$ Lyapunov function V defined in (III.3), we obtain explicitly that $\widehat{\gamma}(r) = \frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}} r$ [28, Proposition 10.5]. We can now present the first main result of the paper, which relates the existence of a δ-ISS-M$_q$ Lyapunov function to the construction of a symbolic model.

Theorem 5.1: Consider a stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f, \sigma)$, admitting a δ-ISS-M$_q$ Lyapunov function V, of the form of (III.3) or the one explained in

Lemmas 3.6. For any $\varepsilon \in \mathbb{R}^+$ and any triple $\mathsf{q} = (\tau, \eta, \mu)$ of quantization parameters satisfying $\mu \leq span(\mathsf{U})$ and

$$\overline{\alpha}\left(\eta^q\right) \leq \underline{\alpha}\left(\varepsilon^q\right), \tag{V.3}$$

$$\mathsf{e}^{-\kappa\tau}\underline{\alpha}\left(\varepsilon^q\right) + \frac{1}{\mathsf{e}\kappa}\rho(\mu) + \widehat{\gamma}\left(\left(h(\sigma,\tau)\right)^{\frac{1}{q}} + \eta\right) \leq \underline{\alpha}\left(\varepsilon^q\right), \tag{V.4}$$

we have that $S_{\mathsf{q}}(\Sigma) \cong_{\mathcal{S}}^{\varepsilon} S_\tau(\Sigma)$. □

It can be readily seen that when we are interested in the dynamics of Σ, initialized on a compact $D \subset \mathbb{R}^n$ of the form of finite union of boxes and for a given precision ε, there always exists a sufficiently large value of τ and small enough values of η and μ, such that $\eta \leq span(D)$ and such that the conditions in (V.3) and (V.4) are satisfied. On the other hand, for a given fixed sampling time τ, one can find sufficiently small values of η and μ satisfying $\eta \leq span(D)$, (V.3) and (V.4), as long as the precision ε is lower bounded by:

$$\varepsilon > \left(\underline{\alpha}^{-1}\left(\frac{\widehat{\gamma}\left(\left(h(\sigma,\tau)\right)^{\frac{1}{q}}\right)}{1-\mathsf{e}^{-\kappa\tau}}\right)\right)^{\frac{1}{q}}. \tag{V.5}$$

One can easily verify that the lower bound on ε in (V.5) goes to zero as τ goes to infinity or as $Z \to 0$, where Z is the Lipschitz constant, introduced in Definition 2.1. Furthermore, one can try to minimize the lower bound on ε in (V.5) by appropriately choosing a δ-ISS-M$_q$ Lyapunov function V (cf. Section III).

The proof of Theorem 5.1 is provided in [30]. Note that the results in [10], as in the following corollary, are fully recovered by the statement in Theorem 5.1 if the stochastic control system Σ is not affected by any noise, implying that $h(\sigma, \tau)$ is identically zero and that the δ-ISS-M$_q$ property reduces to the δ-ISS property.

Corollary 5.2: Let $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f, 0_{n\times p})$ be a δ-ISS control system admitting a δ-ISS Lyapunov function V. For any $\varepsilon \in \mathbb{R}^+$, and any triple $\mathsf{q} = (\tau, \eta, \mu)$ of quantization parameters satisfying $\mu \leq \text{span}(\mathsf{U})$ and

$$\overline{\alpha}\left(\eta\right) \leq \underline{\alpha}(\varepsilon), \tag{V.6}$$

$$\mathsf{e}^{-\kappa\tau}\underline{\alpha}(\varepsilon) + \frac{1}{\mathsf{e}\kappa}\rho(\mu) + \widehat{\gamma}\left(\eta\right) \leq \underline{\alpha}(\varepsilon), \tag{V.7}$$

one obtains $S_{\mathsf{q}}(\Sigma) \cong_{\mathcal{S}}^{\varepsilon} S_\tau(\Sigma)$. □

The next main theorem provides a result that is similar to that in Theorem 5.1, which is however not obtained by explicit use of δ-ISS-M$_q$ Lyapunov functions, but by using functions β and γ as in (III.1).

Theorem 5.3: Consider a δ-ISS-M$_q$ stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f, \sigma)$, satisfying (III.7). For any $\varepsilon \in \mathbb{R}^+$, and any triple $\mathsf{q} = (\tau, \eta, \mu)$ of quantization parameters satisfying $\mu \leq span(\mathsf{U})$ and

$$\left(\beta\left(\varepsilon^q, \tau\right) + \gamma(\mu)\right)^{\frac{1}{q}} + \left(h(\sigma,\tau)\right)^{\frac{1}{q}} + \eta \leq \varepsilon, \tag{V.8}$$

we have $S_{\mathsf{q}}(\Sigma) \cong_{\mathcal{S}}^{\varepsilon} S_\tau(\Sigma)$. □

It can be readily seen that when we are interested in the dynamics of Σ, initialized on a compact $D \subset \mathbb{R}^n$ of the form of finite union of boxes and for a given precision ε, there always exists a sufficiently large value of τ and small values of η and μ such that $\eta \leq span(D)$ and the condition in (V.8) are satisfied. However, unlike the result in Theorem 5.1, notice that here for a given fixed sampling time τ, one may not find any values of η and μ satisfying (V.8) because the quantity $\left(\beta\left(\varepsilon^q, \tau\right)\right)^{\frac{1}{q}}$ may be larger than ε. As long as there exists a triple q, satisfying (V.8), the lower bound of precision ε can be computed by solving the following inequality with respect to ε: $\varepsilon - \left(\beta\left(\varepsilon^q, \tau\right)\right)^{\frac{1}{q}} > \left(h(\sigma,\tau)\right)^{\frac{1}{q}}$. In this case, one can easily verify that the lower bound on ε goes to zero as τ goes to infinity, as τ goes to zero (only if $\left(\beta\left(\varepsilon^q, 0\right)\right)^{\frac{1}{q}} \leq \varepsilon$), or as $Z \to 0$, where Z is the Lipschitz constant introduced in Definition 2.1.

The symbolic model $S_{\mathsf{q}}(\Sigma)$, computed by using the quantization parameters q provided in Theorem 5.3 whenever existing, is likely to have fewer states than the model computed by using the quantization parameters provided in Theorem 5.1, or may provide a better lower bound on ε for a fixed sampling time τ.

The proof of Theorem 5.3 is provided in [30]. Note that the results in [22, Theorem 5.1] are fully recovered by the results in Theorem 5.3 if the stochastic control system Σ is not affected by any noise, implying that $h(\sigma, \tau)$ is identically zero and that the δ-ISS-M$_q$ property becomes the δ-ISS property.

Remark 5.4: Although we assume that the set U is infinite, Theorems 5.1 and 5.3 still hold when the set U is finite, with the following modifications. First, the system Σ is required to satisfy the property (III.1) for $\upsilon = \upsilon'$. Second, take $U_{\mathsf{q}} = \mathsf{U}$ in the definition of $S_{\mathsf{q}}(\Sigma)$. Finally, in the conditions (V.4) and (V.8) set $\mu = 0$. □

VI. EXAMPLE

We now experimentally demonstrate the effectiveness of the discussed results. In the example below, the computation of the abstract system $S_{\mathsf{q}}(\Sigma)$ has been implemented by the software tool Pessoa [19] on a laptop with CPU 2GHz Intel Core i7. We have assumed that the control inputs are piecewise constant of duration τ and that $\mathcal{U}_\tau$ contains curves taking values in $[\mathsf{U}]_\mu$. Hence, as explained in Remark 5.4, $\mu = 0$ is to be used in the conditions (V.4) and (V.8). The controllers enforcing the specifications of interest have been found by standard algorithms from game theory [18], as implemented in Pessoa. In the example, the terms $W_t^i, i = 1, 2$, denote the standard Brownian motion terms.

Consider a linear stochastic control system Σ, borrowed from [1] by adding a control input, and described by:

$$\Sigma : \begin{cases} \mathrm{d}\,\xi_1 = (-2\xi_1 - \xi_2)\,\mathrm{d}\,t + 0.1\xi_1\,\mathrm{d}\,W_t^1, \\ \mathrm{d}\,\xi_2 = (\xi_1 - 0.9\xi_2 + \upsilon)\,\mathrm{d}\,t + 0.1\xi_2\,\mathrm{d}\,W_t^2. \end{cases} \tag{VI.1}$$

It can be readily verified that the system Σ satisfies the condition (III.6) with $\widehat{\kappa} = 2.884$ and the matrix $P = \begin{bmatrix} 2.2101 & 1.2188 \\ 1.2188 & 2.2275 \end{bmatrix}$. Therefore, Σ is δ-ISS-M$_q$, equipped with the δ-ISS-M$_q$ Lyapunov function $V(x, x')$ in (III.3), where $q \in \{1, 2\}$. In this example, we use $q = 1$.

We assume that $\mathsf{U} = [-3,\ 3]$ and $\mathcal{U}_\tau$ contains curves taking values in $[\mathsf{U}]_{0.2}$. Similar to [1], we work on the subset $D = [-2,\ 2] \times [-2,\ 2]$ of the state space of Σ. For a given fixed $\tau = 1.5$, precision ε is lower bounded by 0.1129 and 0.3062 using the results in Theorems 5.3 and 5.1, respectively. Hence, the results in Theorem 5.3 provide a symbolic model which is less conservative than the one provided by Theorem 5.1. By choosing $\varepsilon = 0.12$, the parameter η of $S_{\mathsf{q}}(\Sigma)$ based on the results in Theorem 5.3 is 0.005. The resulting number of states and inputs in $S_{\mathsf{q}}(\Sigma)$ were 641601 and 31, respectively. The consumed CPU time for computing the abstraction was 280.658 seconds.

Now, consider the objective to design a controller forcing (in the 1st moment) the trajectories of Σ to reach and stay within $W = [-0.8,\ -0.7] \times [1.5,\ 1.7]$, while always avoiding $Z = [-1.5,\ -0.5] \times [-1,\ 0.5]$, that is, the LTL specification[1] $\Diamond\Box W \wedge \Box\neg Z$. The consumed CPU time for synthesizing the controller was 77.04 seconds. In Figure 1, we show several realizations of the closed-loop solution process $\xi_{x_0\upsilon}$ stemming from the initial condition $x_0 = (-2, -2)$, as well as the corresponding evolution of the input signal. In Figure 2, we show the average value over 100 experiments of the distance in time of the solution process $\xi_{x_0\upsilon}$ to the sets W and $D\backslash Z$, namely $\|\xi_{x_0\upsilon}(t)\|_W$ and $\|\xi_{x_0\upsilon}(t)\|_{D\backslash Z}$, where the point-to-set distance is defined as $\|x\|_W = \inf_{w\in W} \|x - w\|$. Notice that the average distances are considerably lower than the precision $\varepsilon = 0.12$, as expected since conservative Lyapunov functions can lead to conservative bounds. (As discussed in Corollary 3.5, bounds can be improved by seeking optimized Lyapunov functions.)

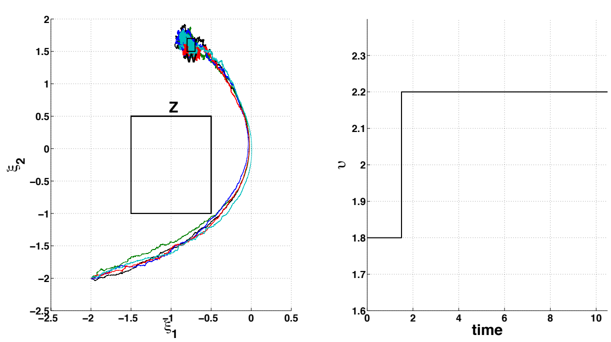

Fig. 1. Example 1: Several realizations of the closed-loop solution process $\xi_{x_0\upsilon}$ with the initial condition $x_0 = (-2, -2)$ (left panel) and evolution of the input signal υ (right panel).

VII. Conclusions

This work has shown that any stochastic sampled-data control system, admitting a δ-ISS-M$_q$ Lyapunov function of the form in (III.3) or with a shape as in Lemma 3.6, and initializing within a compact set of states, admits a finite approximately bisimilar symbolic model (in the sense of moments). The constructed symbolic model can be used to synthesize controllers enforcing complex logic specifications, expressed via linear temporal logic or as automata on infinite strings.

The main limitation of the design methodology developed in this paper lies in the cardinality of the set of states of the computed symbolic model. The authors are currently investigating several different techniques to address this limitation.

[1]Note that the semantics of LTL would be defined over the output behaviors of $S_{\mathsf{q}}(\Sigma)$.

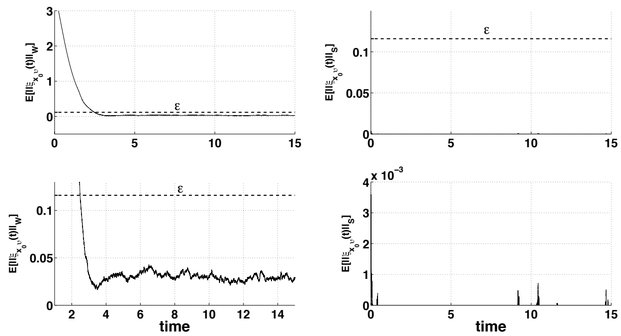

Fig. 2. Example 1: Average values (over 100 experiments) of the distance of the solution process $\xi_{x_0\upsilon}$ to the sets W (left panels) and $S = D\backslash Z$ (right panels), in different vertical scales.

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