Symbolic models for stochastic control systems without stability assumptions

Majid Zamani, Peyman Mohajerin Esfahani, Alessandro Abate, and John Lygeros

***Abstract*— Symbolic approaches provide a mechanism to construct discrete and possibly finite abstractions of continuous control systems. Discrete abstractions are in turn amenable to automata-theoretic techniques targeted to the construction of controllers abiding by complex specifications, which would otherwise be difficult to enforce over continuous models with conventional control design methods. Although construction of discrete abstractions has been extensively studied for deterministic continuous-time control systems, it has received scant attention on stochastic continuous-time non-autonomous models. In this paper, we propose an abstraction technique that is applicable to any stochastic continuous-time control system, as long as we are only interested in its behavior over a compact set. The effectiveness of the proposed results is illustrated with the synthesis of a controller for a jet engine model, which is not stable, affected by noise, and subject to a schedulability constraint expressed by a finite automaton.**

I. INTRODUCTION

Symbolic models are abstract descriptions of physical systems where each state represents a collection, or an aggregate, of states of the continuous system. Symbolic models are as well employed in the description of software and hardware, which are often characterized by discrete, digital components. The composition of continuous and discrete models captures the behavior of physical systems interacting with digital, computational devices, and results in the general framework known as Cyber-Physical Systems (CPS) [22]. The problems of verification and of controller synthesis over models as general as CPS can be algorithmically studied using methodologies and tools developed in the computer science, and particularly in formal methods.

The quest for symbolic abstractions has a rich recent history with numerous results on deterministic continuous control systems [7], [10], [11], [14], [18], [19], [20], [21], [23], [25]. For stochastic systems the results are less abundant, and deal with discrete-time autonomous systems [1], [3], [6], with discrete-time control systems [2] equipped with a finite number of control actions and investigated over reachability analysis, and finally with continuous-time control systems under some stability assumptions [24]. As an extension of [24], this paper shows that a symbolic model of a continuous-time stochastic control system exists even in the absence of any stability assumptions. More specifically, the main contribution of this work is to establish the following claim: *for every continuous-time stochastic nonlinear control system, one can construct a symbolic model that is alternatingly approximately simulated*[1] *by the stochastic control system and that approximately simulates*[2] *the stochastic control system.*

The mentioned relationships are weaker than that of approximate bisimulation relationships established in [24], but they apply to any continuous-time stochastic control system since they no longer require any sort of stability assumptions. Moreover, the relationships established in this paper are still sufficient to guarantee that any controller synthesized for the symbolic model enforces the desired specifications on the original stochastic control system. However, they can no longer guarantee, as it was the case in [24], that the existence of a controller for the original stochastic control system leads to the existence of a controller for the symbolic model.

The technical results in this work are illustrated on a Moore-Greitzer jet engine model, which is affected by noise and dwells in a no-stall mode that does not satisfy the stability assumptions required in [24]. The novel abstraction approach presented in this paper can be used to synthesize a controller stabilizing the jet engine, despite the schedulability constraints imposed by executing the controller actions on a microprocessor running other tasks.

II. STOCHASTIC CONTROL SYSTEMS

A. Notation

The identity map on a set A is denoted by 1_A. If A is a subset of B we denote by $\imath_A : A \hookrightarrow B$ or simply by $\imath$ the natural inclusion map taking any point $a \in A$ to $\imath(a) = a \in B$. The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ denote the set of natural, integer, real, positive real, and nonnegative real numbers, respectively. The symbols I_n, 0_n, and $0_{n\times m}$ denote the identity matrix, the zero vector and zero matrix in $\mathbb{R}^{n\times n}$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times m}$, respectively. Given a vector $x \in \mathbb{R}^n$, we denote by x_i the i–th element of x, and by $\|x\|$ the infinity norm of x, namely, $\|x\| = \max\{|x_1|, |x_2|, ..., |x_n|\}$, where $|x_i|$ denotes the absolute value of x_i. Given a matrix $M = \{m_{ij}\} \in \mathbb{R}^{n\times m}$, we denote by $\|M\|$ the infinity norm of M, namely, $\|M\| = \max_{1\leq i\leq n} \sum_{j=1}^{m} |m_{ij}|$, and by $\|M\|_F$ the Frobenius norm of M, namely, $\|M\|_F = \sqrt{\text{Tr}\,(MM^T)}$ where Tr denotes the trace.

The closed ball centered at $x \in \mathbb{R}^n$ with radius ε is defined by $\mathcal{B}_\varepsilon(x) = \{y \in \mathbb{R}^n \mid \|x - y\| \leq \varepsilon\}$. A set $B \subseteq \mathbb{R}^n$ is called a *box* if $B = \prod_{i=1}^n [c_i, d_i]$, where $c_i, d_i \in \mathbb{R}$ with $c_i < d_i$ for each $i \in \{1, \ldots, n\}$. The *span* of a box B is defined as $span(B) = \min\{|d_i - c_i| \mid i = 1, \ldots, n\}$. For a box B and $\eta \leq span(B)$, define the η-approximation $[B]_\eta = \{b \in B \mid b_i = k_i\eta \text{ for some } k_i \in \mathbb{Z}, i = 1, \ldots, n\}$. Note that $[B]_\eta \neq \varnothing$ for any $\eta \leq span(B)$. Geometrically, for any $\eta \in \mathbb{R}^+$ with $\eta \leq span(B)$ and $\lambda \geq \eta$, the collection of sets $\{\mathcal{B}_\lambda(p)\}_{p\in[B]_\eta}$ is a finite covering of B, i.e., $B \subseteq \bigcup_{p\in[B]_\eta} \mathcal{B}_\lambda(p)$. By defining $[\mathbb{R}^n]_\eta = \{a \in \mathbb{R}^n \mid a_i = k_i\eta, k_i \in \mathbb{Z}, i = 1, \cdots, n\}$, the set $\bigcup_{p\in[\mathbb{R}^n]_\eta} \mathcal{B}_\lambda(p)$ is a countable covering of $\mathbb{R}^n$ for

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M. Zamani and A. Abate are with the Delft Center for Systems and Control, Delft University of Technology, 2628 CD Delft, the Netherlands. Email: {m.zamani,a.abate}@tudelft.nl, URL: http://www.dcsc.tudelft.nl/∼{mzamani,aabate}.

P. Mohajerin Esfahani and J. Lygeros are with the Automatic Control Laboratory, ETH Zurich, ETL I22, 8092 Zurich, Switzerland. Email: {mohajerin,lygeros}@control.ee.ethz.ch, URL: http://control.ee.ethz.ch/∼{peymanm,jlygeros}.

[1] As defined in Definition 3.3.

[2] As defined in Definition 3.2.

any $\eta \in \mathbb{R}^+$ and $\lambda \geq \eta$. We extend the notions of $span$ and of approximation to finite unions of boxes as follows. Let $A = \bigcup_{j=1}^{M} A_j$, where each A_j is a box. Define $span(A) = \min\{span(A_j) \mid j = 1, \ldots, M\}$, and for any $\eta \leq span(A)$, define $[A]_\eta = \bigcup_{j=1}^{M}[A_j]_\eta$.

Given a measurable function $f : \mathbb{R}_0^+ \to \mathbb{R}^n$, the (essential) supremum (sup norm) of f is denoted by $\|f\|_\infty$; we recall that $\|f\|_\infty = (\text{ess})\sup\{\|f(t)\|, t \geq 0\}$. A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. We identify a relation $R \subseteq A \times B$ with the map $R : A \to 2^B$ defined by $b \in R(a)$ iff $(a, b) \in R$. Given a relation $R \subseteq A \times B$, R^{-1} denotes the inverse relation defined by $R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$.

B. Digital stochastic control systems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_s)_{s \geq 0}$ satisfying usual conditions of completeness and right continuity [12, p. 48]. Let $(W_s)_{s \geq 0}$ be a q-dimensional $\mathbb{F}$-Brownian motion.

Definition 2.1: A *(digital) stochastic control system* is a tuple $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f, \sigma)$, where

- $\mathbb{R}^n$ is the state space;
- $\mathsf{U} \subseteq \mathbb{R}^m$ is the input set, which is assumed to be a finite union of boxes;
- $\mathcal{U}_\tau$ contains piecewise constant curves of duration τ:

$$\mathcal{U}_\tau = \Big\{ \upsilon : \mathbb{R}_0^+ \to \mathsf{U} \mid \upsilon(t) = \upsilon((l-1)\tau), \\ t \in [(l-1)\tau, l\tau[,\ l \in \mathbb{N} \Big\};$$

- $f : \mathbb{R}^n \times \mathsf{U} \to \mathbb{R}^n$ is a continuous function of its arguments satisfying the following Lipschitz assumption: for all $x, x' \in \mathbb{R}^n$ and all $u, u' \in \mathsf{U}$, there exist constants $L_x, L_u \in \mathbb{R}^+$ such that: $\|f(x, u) - f(x', u')\| \leq L_x\|x - x'\| + L_u\|u - u'\|$;
- $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times q}$ is a function satisfying the following Lipschitz assumption: for all $x, x' \in \mathbb{R}^n$, there exists a constant $Z \in \mathbb{R}^+$ such that: $\|\sigma(x) - \sigma(x')\| \leq Z\|x - x'\|$.

Note that the results and definitions in Section II and III are still valid even if $\mathcal{U}_\tau$ allows for any measurable and locally essentially bounded curve of time. Furthermore, the results in Section IV can still be shown even if $\mathcal{U}_\tau$ contains curves that are Lipschitz continuous in each interval of duration τ. However, from the point of view of abstractions and refinements dealt with in this work, it is natural to directly handle piecewise constant curves.

A stochastic process $\xi : \Omega \times [0, \infty[\to \mathbb{R}^n$ is said to be a *solution process* of Σ if there exists $\upsilon \in \mathcal{U}_\tau$ satisfying:

$$\mathrm{d}\,\xi = f(\xi, \upsilon)\,\mathrm{d}\,t + \sigma(\xi)\,\mathrm{d}\,W_t, \tag{II.1}$$

$\mathbb{P}$-almost surely ($\mathbb{P}$-a.s.). We also write $\xi_{a\upsilon}(t)$ to denote the value of the solution process at time $t \in \mathbb{R}_0^+$ under the input υ and from the initial condition $\xi_{a\upsilon}(0) = a$ $\mathbb{P}$-a.s., in which a is a random variable that is measurable in $\mathcal{F}_0$. Note that $\mathcal{F}_0$, in general, is not a trivial sigma-algebra, and stochastic control system Σ may start from a random initial condition. Let us emphasize that this solution process is uniquely determined, up to indistinguishability, since the assumptions on f and σ ensure the existence and the uniqueness of solutions [16, Theorem 5.2.1, p. 68].

C. Stochastic incremental forward completeness

The results presented in this paper require a certain property on Σ that we introduce in this section.

Definition 2.2: A stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f, \sigma)$ is stochastically incrementally forward complete (δ-SFC) if there exist continuos functions $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and $\gamma : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that for every $s \in \mathbb{R}^+$, the functions $\beta(\cdot, s)$ and $\gamma(\cdot, s)$ belong to class $\mathcal{K}_\infty$, and for any $\mathbb{R}^n$-valued random variables a and a', which are measurable in $\mathcal{F}_0$, any $t \in \mathbb{R}^+$, and any $\upsilon, \upsilon' \in \mathcal{U}_\tau$, the following condition is satisfied:

$$\mathbb{E}\left[\|\xi_{a\upsilon}(t) - \xi_{a'\upsilon'}(t)\|\right] \leq \beta\left(\mathbb{E}\left[\|a - a'\|\right], t\right) + \gamma(\|\upsilon - \upsilon'\|_\infty, t). \tag{II.2}$$

The result in Theorem 2.4 will show that any stochastic control system Σ is indeed δ-SFC. The notion of δ-SFC can be characterized in terms of Lyapunov-like functions. It will be shown later that the proposed Lyapunov-like functions can be chosen appropriately by solving a matrix inequality to get a tighter upper bound in (II.2). We start by introducing the following definition, which is inspired by the notion of incrementally forward complete (δ-FC) Lyapunov function presented in [25] for deterministic models.

Definition 2.3: Consider a stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f, \sigma)$ and a continuous function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ that is smooth on $\{\mathbb{R}^n \times \mathbb{R}^n\} \backslash \Delta$.[3] Function V is called a *stochastic incrementally forward complete* (δ-SFC) Lyapunov function for Σ if there exist $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, ρ, and a constant $\kappa \in \mathbb{R}$ such that

(i) $\underline{\alpha}$ (resp. $\overline{\alpha}$) is a convex (resp. concave) function;
(ii) $\forall x, x' \in \mathbb{R}^n$, $\underline{\alpha}(\|x - x'\|) \leq V(x, x') \leq \overline{\alpha}(\|x - x'\|)$;
(iii) for any $x, x' \in \mathbb{R}^n : x \neq x'$, and for any $u, u' \in \mathsf{U}$,

$$\begin{aligned}\mathcal{L}^{u,u'}V(x, x') &:= \begin{bmatrix}\partial_x V & \partial_{x'} V\end{bmatrix} \begin{bmatrix} f(x, u) \\ f(x', u') \end{bmatrix} \\ &+ \frac{1}{2}\mathrm{Tr}\left(\begin{bmatrix}\sigma(x) \\ \sigma(x')\end{bmatrix}\begin{bmatrix}\sigma^T(x) & \sigma^T(x')\end{bmatrix}\begin{bmatrix}\partial_{x,x}V & \partial_{x,x'}V \\ \partial_{x',x}V & \partial_{x',x'}V\end{bmatrix}\right) \\ &\leq \kappa V(x, x') + \rho(\|u - u'\|),\end{aligned}$$

where $\mathcal{L}^{u,u'}$ is the infinitesimal generator associated to the stochastic control system (II.1) [16, Section 7.3], which in this case depends on two separate controls u, u'.

Note that the condition (i) is not required in the context of deterministic control systems in [25]. Roughly speaking, condition (ii) implies that the growth rate of functions $\underline{\alpha}$ and $\overline{\alpha}$ are linear, as a concave function is supposed to dominate a convex one. These conditions are not restrictive, provided we are interested in the dynamics of Σ on a compact subset $D \subset \mathbb{R}^n$, which is often the case in practice.

Theorem 2.4: A stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f, \sigma)$ is δ-SFC if and only if it admits a δ-SFC Lyapunov function.

Proof: We first prove the sufficient part of the proof. The proof is a consequence of applications of Gronwall's inequality and Ito's lemma [16, p. 80 and 123]. Assume there exists a δ-SFC Lyapunov function in the sense of Definition 2.3. For any $t \in \mathbb{R}_0^+$, any $\upsilon, \upsilon' \in \mathcal{U}_\tau$, and any $\mathbb{R}^n$-valued

[3] $\Delta = \left\{\chi \in \mathbb{R}^n \times \mathbb{R}^n \mid \chi = \left[x^T, x^T\right]^T, x \in \mathbb{R}^n\right\}$ is a diagonal set.

random variables a and a', measurable in $\mathcal{F}_0$, we obtain

$$\begin{aligned}
&\mathbb{E}\left[V(\xi_{av}(t),\xi_{a'v'}(t))\right] = \\
&\mathbb{E}\left[V(a,a') + \int_0^t \mathcal{L}^{v(s),v'(s)}V(\xi_{av}(s),\xi_{a'v'}(s))ds\right] \leq \\
&\mathbb{E}\left[V(a,a') + \int_0^t \left(\kappa V(\xi_{av}(s),\xi_{a'v'}(s)) + \rho(\|v(s)-v'(s)\|)\right)ds\right] \\
&\leq \kappa \int_0^t \mathbb{E}\left[V(\xi_{av}(s),\xi_{a'v'}(s))\right]ds + \mathbb{E}[V(a,a')] + \rho(\|v-v'\|_\infty)t,
\end{aligned}$$

where, by virtue of Gronwall's inequality, it leads to

$$\mathbb{E}\left[V(\xi_{av}(t),\xi_{a'v'}(t))\right] \leq \mathbb{E}[V(a,a')]\mathsf{e}^{\kappa t} + t\mathsf{e}^{\kappa t}\rho(\|v-v'\|_\infty). \tag{II.3}$$

Hence, using property (ii) in Definition 2.3, we have

$$\begin{aligned}
&\underline{\alpha}\left(\mathbb{E}[\|\xi_{av}(t)-\xi_{a'v'}(t)\|]\right) \leq \mathbb{E}\left[\underline{\alpha}(\|\xi_{av}(t)-\xi_{a'v'}(t)\|)\right] \\
&\leq \mathbb{E}\left[V(\xi_{av}(t),\xi_{a'v'}(t))\right] \\
&\leq \mathbb{E}[V(a,a')]\mathsf{e}^{\kappa t} + t\mathsf{e}^{\kappa t}\rho(\|v-v'\|_\infty) \\
&\leq \mathbb{E}\left[\overline{\alpha}\big(\|a-a'\|\big)\right]\mathsf{e}^{\kappa t} + t\mathsf{e}^{\kappa t}\rho(\|v-v'\|_\infty) \\
&\leq \overline{\alpha}\big(\mathbb{E}\left[\|a-a'\|\right]\big)\mathsf{e}^{\kappa t} + t\mathsf{e}^{\kappa t}\rho(\|v-v'\|_\infty),
\end{aligned} \tag{II.4}$$

where the first and last inequalities follow from property (i) and the Jensen inequality [16, p. 310]. Since $\underline{\alpha} \in \mathcal{K}_\infty$, the inequality (II.4) yields

$$\begin{aligned}
&\mathbb{E}\left[\|\xi_{av}(t)-\xi_{a'v'}(t)\|\right] \\
&\leq \underline{\alpha}^{-1}\left(\overline{\alpha}\left(\mathbb{E}\left[\|a-a'\|\right]\right)\mathsf{e}^{\kappa t} + t\mathsf{e}^{\kappa t}\rho(\|v-v'\|_\infty)\right) \\
&\leq \underline{\alpha}^{-1}\left(\overline{\alpha}\left(\mathbb{E}\left[\|a-a'\|\right]\right)\mathsf{e}^{\kappa t} + \overline{\alpha}\left(\mathbb{E}\left[\|a-a'\|\right]\right)\mathsf{e}^{\kappa t}\right) \\
&\quad + \underline{\alpha}^{-1}\left(t\mathsf{e}^{\kappa t}\rho(\|v-v'\|_\infty) + t\mathsf{e}^{\kappa t}\rho(\|v-v'\|_\infty)\right) \\
&\leq \underline{\alpha}^{-1}\left(2\overline{\alpha}\left(\mathbb{E}\left[\|a-a'\|\right]\right)\mathsf{e}^{\kappa t}\right) + \underline{\alpha}^{-1}\left(2t\mathsf{e}^{\kappa t}\rho(\|v-v'\|_\infty)\right).
\end{aligned} \tag{II.5}$$

Therefore, by introducing functions β and γ as

$$\begin{aligned}
\beta\left(\mathbb{E}\left[\|a-a'\|\right],t\right) &:= \underline{\alpha}^{-1}\left(2\overline{\alpha}\left(\mathbb{E}\left[\|a-a'\|\right]\right)\mathsf{e}^{\kappa t}\right), \\
\gamma\left(\|v-v'\|_\infty,t\right) &:= \underline{\alpha}^{-1}\left(2t\mathsf{e}^{\kappa t}\rho(\|v-v'\|_\infty)\right),
\end{aligned}$$

the condition (II.2) is satisfied. Hence, the system Σ is δ-SFC.

Now, we prove the necessary part of the proof by showing that any stochastic control system Σ admits a δ-SFC Lyapunov function as $V(x,x') = \sqrt{(x-x')^T(x-x')}$. It is not difficult to check that the function V satisfies properties (i) and (ii) of Definition 2.3 with functions $\underline{\alpha}(y) := y$ and $\overline{\alpha}(y) := \sqrt{n}y$. It then suffices to verify property (iii). By the definition of V, for any $x,x' \in \mathbb{R}^n$ such that $x \neq x'$, one has

$$\begin{aligned}
\partial_x V = -\partial_{x'}V &= \frac{(x-x')^T}{V(x,x')}, \\
\partial_{x,x}V = \partial_{x',x'}V &= -\partial_{x,x'}V \\
&= \frac{V^2(x,x')I_n - (x-x')(x-x')^T}{V^3(x,x')}.
\end{aligned}$$

Therefore, following the definition of $\mathcal{L}^{u,u'}$, for any $x,x' \in \mathbb{R}^n$ such that $x \neq x'$, and any $u,u' \in \mathsf{U}$, one obtains

$$\begin{aligned}
&\mathcal{L}^{u,u'}V(x,x') = \frac{(x-x')^T}{V(x,x')}\left(f(x,u)-f(x',u')\right) \\
&\quad + \frac{1}{2}\mathrm{Tr}\left(\begin{bmatrix}\sigma(x)\\ \sigma(x')\end{bmatrix}\begin{bmatrix}\sigma^T(x) & \sigma^T(x')\end{bmatrix}\begin{bmatrix}\partial_{x,x}V & -\partial_{x,x}V\\ -\partial_{x,x}V & \partial_{x,x}V\end{bmatrix}\right) \\
&= \frac{(x-x')^T}{V(x,x')}\left(f(x,u)-f(x',u')\right) \\
&\quad + \frac{1}{2}\mathrm{Tr}\left(\left(\sigma(x)-\sigma(x')\right)\left(\sigma^T(x)-\sigma^T(x')\right)\partial_{x,x}V\right) \\
&= \frac{(x-x')^T}{V(x,x')}\left(f(x,u)-f(x',u')\right) \\
&\quad + \frac{1}{2V^3(x,x')}\left(\left\|\left(\sigma(x)-\sigma(x')\right)\right\|_F^2 V^2(x,x')\right. \\
&\quad \left. - \left\|(x-x')^T\left(\sigma(x)-\sigma(x')\right)\right\|_F^2\right) \\
&\leq \frac{\|x-x'\|}{V(x,x')}\left(L_x\|x-x'\| + L_u\|u-u'\|\right) + \frac{\|\sigma(x)-\sigma(x')\|_F^2}{2V(x,x')} \\
&\leq L_x\|x-x'\| + L_u\|u-u'\| + \frac{\min\{n,q\}nZ^2\|x-x'\|^2}{2V(x,x')} \\
&\leq \left(L_x + \frac{\min\{n,q\}nZ^2}{2}\right)V(x,x') + L_u\left\|u-u'\right\|,
\end{aligned}$$

where L_x, L_u, and Z are the Lipschitz constants, as introduced in Definition 2.1. Therefore, $V(x,x') = \sqrt{(x-x')^T(x-x')}$ is a δ-SFC Lyapunov function for Σ. ■

The following result provides a sufficient condition on a particular function V to be a δ-SFC Lyapunov function.

Lemma 2.5: Consider a stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f, \sigma)$. Let $P \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix, and the function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ be defined as follows:

$$V(x,x') := \sqrt{(x-x')^T P (x-x')}, \tag{II.6}$$

and satisfy

$$(x-x')^T P(f(x,u)-f(x',u)) + \frac{1}{2}\left\|\sqrt{P}\left(\sigma(x)-\sigma(x')\right)\right\|_F^2 \leq \kappa V^2(x,x'), \tag{II.7}$$

or, if f is differentiable, let V satisfy

$$(x-x')^T P\partial_x f(z,u)(x-x') + \frac{1}{2}\left\|\sqrt{P}\left(\sigma(x)-\sigma(x')\right)\right\|_F^2 \leq \kappa V^2(x,x'), \tag{II.8}$$

for all x,x',z in $\mathbb{R}^n$, for all $u \in \mathsf{U}$, and for some constant $\kappa \in \mathbb{R}$. Then V is a δ-SFC Lyapunov function for Σ.

Proof: It is not difficult to check that the function V in (II.6) satisfies properties (i) and (ii) of Definition 2.3 with functions $\underline{\alpha}(y) := \sqrt{\lambda_{\min}(P)}y$ and $\overline{\alpha}(y) := \sqrt{n\lambda_{\max}(P)}y$, where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are minimum and maximum eigenvalues of P, respectively. Property (iii) can be readily verified similarly to the second part of the proof of Theorem 2.4 and using the mean value theorem [8] applied to the differentiable function $x \mapsto f(x,u)$, for a given input value $u \in \mathsf{U}$ and for point z within x and x'. ■

The next result provides an equivalent condition to (II.7) for linear stochastic control systems in the form of a linear matrix inequality (LMI).

Corollary 2.6: Consider a stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f, \sigma)$, where $f(x,u) := Ax + Bu$ for any $x \in \mathbb{R}^n$ and any $u \in \mathsf{U}$, and

$\sigma(x) := [\sigma_1 x \quad \sigma_2 x \quad \cdots \quad \sigma_q x]$, where $\sigma_i \in \mathbb{R}^{n\times n}$. Then, the function V in (II.6) is a δ-SFC Lyapunov function for Σ if there exists a constant $\widehat{\kappa} \in \mathbb{R}$ satisfying the following LMI:

$$PA + A^T P + \sum_{i=1}^{q} \sigma_i^T P \sigma_i \prec \widehat{\kappa} P. \tag{II.9}$$

Proof: The corollary is a particular case of Lemma 2.5. It suffices to show that for linear dynamics the LMI (II.9) yields to the condition (II.7). First it is straightforward to observe that

$$\left\|\sqrt{P}\left(\sigma(x)-\sigma(x')\right)\right\|_F^2 = \mathrm{Tr}\left(\left(\sigma(x)-\sigma(x')\right)^T P \left(\sigma(x)-\sigma(x')\right)\right)$$
$$= \left(x-x'\right)^T \sum_{i=1}^{q} \sigma_i^T P \sigma_i (x-x'),$$

and that

$$(x-x')^T P(f(x,u)-f(x',u)) = \frac{1}{2}(x-x')^T \left(PA + A^T P\right)(x-x'),$$

for any $x, x', z \in \mathbb{R}^n$ and any $u \in \mathsf{U}$. Now suppose there exists $\widehat{\kappa} \in \mathbb{R}$ such that (II.9) holds. It can then be verified that the assertion in (II.7) holds by choosing $\kappa = \widehat{\kappa}/2$. ■

Hence, one can find an appropriate matrix P by solving the LMI (II.9) to have a tighter upper bound in (II.2).

III. Symbolic Models and Approximate Equivalence Notions

A. Systems

We use systems to describe both stochastic control systems as well as their symbolic models.

Definition 3.1: [22] A system S is a tuple $S = (X, X_0, U, \longrightarrow, Y, H)$ consisting of

- A set of states X;
- A set of initial states $X_0 \subseteq X$;
- A set of inputs U;
- A transition relation $\longrightarrow \subseteq X \times U \times X$;
- An output set Y;
- An output function $H : X \to Y$.

A system S is said to be

- *metric*, if the output set Y is equipped with a metric $\mathbf{d} : Y \times Y \to \mathbb{R}_0^+$;
- *finite*, if X is a finite set.

A transition $(x, u, x') \in \longrightarrow$ is also denoted by $x \xrightarrow{u} x'$. For a transition $x \xrightarrow{u} x'$, state x' is called a u-successor, or simply a successor, of state x. We denote by $\mathbf{Post}_u(x)$ the set of u-successors of a state x and by $U(x)$ the set of inputs $u \in U$ for which $\mathbf{Post}_u(x)$ is nonempty. A system is deterministic if for any state $x \in X$ and any input u, there exists at most one u-successor (there may be none). A system is called nondeterministic if it is not deterministic. Hence, for a nondeterministic system it is possible for a state to have two (or possibly more) distinct u-successors.

B. Relations among systems

First, we recall the notion of approximate simulation relation, introduced in [9], which is useful when analyzing or synthesizing controllers for deterministic systems.

Definition 3.2: Let $S_a = (X_a, X_{a0}, U_a, \underset{a}{\longrightarrow}, Y_a, H_a)$ and $S_b = (X_b, X_{b0}, U_b, \underset{b}{\longrightarrow}, Y_b, H_b)$ be metric systems with the same output sets $Y_a = Y_b$ and metric $\mathbf{d}$, and consider a precision $\varepsilon \in \mathbb{R}^+$. A relation $R \subseteq X_a \times X_b$ is said to be an ε-approximate simulation relation from S_a to S_b, if the following three conditions are satisfied:

(i) $\forall x_{a0} \in X_{a0}, \exists x_{b0} \in X_{b0}$ with $(x_{a0}, x_{b0}) \in R$;
(ii) $\forall (x_a, x_b) \in R$ we have $\mathbf{d}(H_a(x_a), H_b(x_b)) \le \varepsilon$;
(iii) $\forall (x_a, x_b) \in R$, $x_a \underset{a}{\xrightarrow{u_a}} x'_a$ in S_a implies the existence of $x_b \underset{b}{\xrightarrow{u_b}} x'_b$ in S_b satisfying $(x'_a, x'_b) \in R$.

System S_a is ε-approximately simulated by S_b or S_b ε-approximately simulates S_a, denoted by $S_a \preceq_{\mathcal{S}}^{\varepsilon} S_b$, if there exists an ε-approximate simulation relation from S_a to S_b.

Note that when $\varepsilon = 0$, condition (ii) in the above definition is changed to: $(x_a, x_b) \in R \Leftrightarrow H_a(x_a) = H_b(x_b)$; and R becomes an exact simulation relation as in [15].

For nondeterministic systems we need to consider relationships that explicitly capture the adversarial nature of nondeterminism. The notion of alternating approximate simulation relation is shown in [20] to be appropriate for this.

Definition 3.3: Let $S_a = (X_a, X_{a0}, U_a, \underset{a}{\longrightarrow}, Y_a, H_a)$ and $S_b = (X_b, X_{b0}, U_b, \underset{b}{\longrightarrow}, Y_b, H_b)$ be metric systems with the same output sets $Y_a = Y_b$ and metric $\mathbf{d}$, and consider a precision $\varepsilon \in \mathbb{R}^+$. A relation $R \subseteq X_a \times X_b$ is said to be an alternating ε-approximate simulation relation from S_a to S_b if conditions (i), (ii) in Definition 3.2, and additionally the following condition, are satisfied:

(iii) for every $(x_a, x_b) \in R$ and for every $u_a \in U_a(x_a)$ there exists $u_b \in U_b(x_b)$ such that for every $x'_b \in \mathbf{Post}_{u_b}(x_b)$ there exists $x'_a \in \mathbf{Post}_{u_a}(x_a)$ satisfying $(x'_b, x'_a) \in R$.

System S_a is alternatingly ε-approximately simulated by S_b or S_b alternatingly ε-approximately simulates S_a, denoted by $S_a \preceq_{\mathcal{AS}}^{\varepsilon} S_b$, if there exists an alternating ε-approximate simulation relation from S_a to S_b.

Note that when $\varepsilon = 0$, condition (ii) in the above definition is changed to: $(x_a, x_b) \in R \Leftrightarrow H_a(x_a) = H_b(x_b)$; and R becomes an exact alternating simulation relation, as introduced in [4]. It is readily seen from the above definitions that the notions of approximate simulation and of alternating approximate simulation coincide when the systems involved are deterministic.

IV. Symbolic Models for Stochastic Control Systems

This section contains the main contribution of the paper. We show that any stochastic control system Σ admits a finite symbolic model whenever we are interested in the dynamics of Σ on a compact subset of $\mathbb{R}^n$. The results in this section rely on the additional assumptions $f(0_n, 0_m) = 0_n$, and $\sigma(0_n) = 0_{n\times q}$, used in Lemma 4.1.

Given a stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f, \sigma)$, consider the system

$$S_\tau(\Sigma) = (X_\tau, X_{\tau 0}, U_\tau, \underset{\tau}{\longrightarrow}, Y_\tau, H_\tau),$$

consisting of:

- X_τ is the set of all $\mathbb{R}^n$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- $X_{\tau 0}$ is the set of random variable measurable with respect to trivial sigma-algebra $\mathcal{F}_0$, i.e., the system starts from a deterministic initial condition;
- $U_\tau = \mathcal{U}_\tau$;
- $x_\tau \underset{\tau}{\xrightarrow{\upsilon_\tau}} x'_\tau$ if x_τ and x'_τ are measurable, respectively, in $\mathcal{F}_{k\tau}$ and $\mathcal{F}_{(k+1)\tau}$ for some $k \in \mathbb{N} \cup \{0\}$, and there exists a solution process $\xi : \mathbb{R}^+ \to \mathbb{R}^n$ of Σ satisfying $\xi(k\tau) = x_\tau$ and $\xi_{x_\tau \upsilon_\tau}(\tau) = x'_\tau$ $\mathbb{P}$-a.s.;

- Y_τ is the set of all $\mathbb{R}^n$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- $H_\tau = 1_{X_\tau}$.

We assume that the output set Y_τ is equipped with the metric $\mathbf{d}(y, y') = \mathbb{E}\left[\|y - y'\|\right]$, for any $y, y' \in Y_\tau$.

Before introducing the symbolic model for the stochastic control system, we proceed with the following preliminary lemma.

Lemma 4.1: Consider a stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f, \sigma)$. Suppose that a function V in (II.6) satisfies (II.7) or (II.8) for Σ. For any $x \in \mathbb{R}^n$ and any $\upsilon \in \mathcal{U}_\tau$, we have

$$\mathbb{E}\left[\|\xi_{x\upsilon}(t) - \overline{\xi}_{x\upsilon}(t)\|\right] \leq h(t, \sigma)\mathsf{e}^{\kappa t}, \qquad \text{(IV.1)}$$

where κ is the same constant introduced in (II.7) or (II.8), $\overline{\xi}_{x\upsilon}$ is the solution of the ordinary differential equation (ODE) $\dot{\overline{\xi}}_{x\upsilon}(t) = f\left(\overline{\xi}_{x\upsilon}(t), \upsilon(t)\right)$ starting from the initial condition x, and the nonnegative valued function h tends to zero as $t \to 0$ or as $\sup_x\{\|\sigma(x)\|\} \to 0$.

The proof of Lemma 4.1 is provided in Appendix.

Remark 4.2: In case of a linear stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f, \sigma)$, where $f(x, u) := Ax + Bu$ and $\sigma(x) := [\sigma_1 x \quad \sigma_2 x \quad \cdots \quad \sigma_q x]$ with $\sigma_i \in \mathbb{R}^{n \times n}$, one may deduce a tighter and in fact explicit bound in the previous lemma as follows. Motivated by inequality (V.2) in the Appendix, one can obtain

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{Tr}\left(\sigma\sigma^T\left(\xi_{x\upsilon}(s)\right)P - \sigma\sigma^T\left(\xi_{x\upsilon}(s) - \overline{\xi}_{x\upsilon}(s)\right)P\right)\right] \\
&= \mathbb{E}\Bigg[\xi_{x\upsilon}^T(s)\left(\sum_{i=1}^q \sigma_i^T P \sigma_i\right)\xi_{x\upsilon}(s) \\
&\qquad - \left(\xi_{x\upsilon}(s) - \overline{\xi}_{x\upsilon}(s)\right)^T\left(\sum_{i=1}^q \sigma_i^T P \sigma_i\right)\left(\xi_{x\upsilon}(s) - \overline{\xi}_{x\upsilon}(s)\right)\Bigg] \\
&= \overline{\xi}_{x\upsilon}^T(s)\left(\sum_{i=1}^q \sigma_i^T P \sigma_i\right)\overline{\xi}_{x\upsilon}(s) \leq n\lambda(\sigma, P)\left\|\overline{\xi}_{x\upsilon}(s)\right\|^2, \qquad \text{(IV.2)}
\end{aligned}$$

where $\lambda(\sigma, P)$ is the maximum eigenvalue of $\sum_{i=1}^q \sigma_i^T P \sigma_i$, and $\overline{\xi}_{x\upsilon}$ satisfies the ODE $\dot{\overline{\xi}}_{x\upsilon}(t) = A\overline{\xi}_{x\upsilon}(t) + B\upsilon(t)$. It can be readily verified that

$$\begin{aligned}
\left\|\overline{\xi}_{x\upsilon}(t)\right\| &\leq \left\|\mathsf{e}^{At}\right\|\|x\| + \left(\int_0^t \left\|\mathsf{e}^{As}B\right\| ds\right)\|\upsilon\|_\infty \qquad \text{(IV.3)} \\
&\leq C_1\mathsf{e}^{\lambda_{\max} t}\|x\| + \frac{C_2}{\lambda_{\max}}\left(\mathsf{e}^{\lambda_{\max} t} - 1\right)\|\upsilon\|_\infty,
\end{aligned}$$

for some constants $C_1, C_2 \in \mathbb{R}^+$, and $\lambda_{\max}$ is the maximum real part among all eigenvalues of A. The above approximation, together with (IV.2) and (V.2), leads to an explicit bound in terms of the system parameters as follows:

$$h(t, \sigma) = \sqrt{\frac{n\lambda(\sigma, P)}{\lambda_{\min}(P)}} \cdot \sqrt{\int_0^t \left(C_1\mathsf{e}^{\lambda_{\max} s}\|x\| + \frac{C_2}{\lambda_{\max}}\left(\mathsf{e}^{\lambda_{\max} s} - 1\right)\|\upsilon\|_\infty\right)^2 ds}. \qquad \text{(IV.4)}$$

Note that when we are interested in the dynamics of Σ on a compact subset $D \subset \mathbb{R}^n$, one gets the following upper bound:

$$h(t, \sigma) = \sqrt{\frac{n\lambda(\sigma, P)}{\lambda_{\min}(P)}} \cdot \sqrt{\int_0^t \left(C_1\mathsf{e}^{\lambda_{\max} s}\sup_{x \in D}\{\|x\|\} + \frac{C_2}{\lambda_{\max}}\left(\mathsf{e}^{\lambda_{\max} s} - 1\right)\sup_{u \in \mathsf{U}}\{\|u\|\}\right) ds}. \qquad \text{(IV.5)}$$

We consider a stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f, \sigma)$, and a tuple $\mathsf{q} = (\tau, \eta, \mu, \theta, k)$ of quantization parameters, where $\tau \in \mathbb{R}^+$ is the sampling time, $\eta \in \mathbb{R}^+$ is the state space quantization, $\mu \in \mathbb{R}^+$ is the input set quantization and $\theta \in \mathbb{R}^+$ and $\ell \in \mathbb{N}$ are design parameters. Given Σ and q, consider the following system:

$$S_{\mathsf{q}}(\Sigma) = (X_{\mathsf{q}}, X_{\mathsf{q}0}, U_{\mathsf{q}}, \underset{\mathsf{q}}{\longrightarrow}, Y_{\mathsf{q}}, H_{\mathsf{q}}), \qquad \text{(IV.6)}$$

consisting of:

- $X_{\mathsf{q}} = [\mathbb{R}^n]_\eta$;
- $X_{\mathsf{q}0} = [\mathbb{R}^n]_\eta$;
- $U_{\mathsf{q}} = [\mathsf{U}]_\mu$;
- $x_{\mathsf{q}} \xrightarrow[\mathsf{q}]{u_{\mathsf{q}}} x'_{\mathsf{q}}$ if $\left\|\overline{\xi}_{x_{\mathsf{q}}u_{\mathsf{q}}}(\tau) - x'_{\mathsf{q}}\right\| \leq \beta(\theta, \tau) + \gamma(\mu, \tau) + h(\tau, \sigma)\mathsf{e}^{\kappa\tau} + h(\ell\tau, \sigma)\mathsf{e}^{\kappa\ell\tau} + \eta$, where $\dot{\overline{\xi}}_{x_{\mathsf{q}}u_{\mathsf{q}}}(t) = f\left(\overline{\xi}_{x_{\mathsf{q}}u_{\mathsf{q}}}(t), u_{\mathsf{q}}(t)\right)$;
- Y_{q} is the set of all $\mathbb{R}^n$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- $H_{\mathsf{q}} = \imath : X_{\mathsf{q}} \hookrightarrow Y_{\mathsf{q}}$.

Here β and γ are the functions appearing in (II.2) and h and κ are the function and constant appearing in (IV.1). Note that we have abused notation by identifying $u_{\mathsf{q}} \in [\mathsf{U}]_\mu$ with the constant input curve with domain $[0, \tau[$ and value u_{q}. Notice that the proposed abstraction $S_{\mathsf{q}}(\Sigma)$ is indeed a nondeterministic system governed by an ordinary differential equation. However, in order to establish an (alternating) approximate simulation relation, the output set Y_{q} is defined similarly to our original stochastic system $S_\tau(\Sigma)$. Therefore, in the definition of H_{q}, the inclusion map $\imath$ is meant, with a slight abuse of notation, a mapping from a deterministic grid point to a random variable with a Dirac probability distribution centered at the grid point.

The transition relation of $S_{\mathsf{q}}(\Sigma)$ is well defined in the sense that for every $x_{\mathsf{q}} \in [\mathbb{R}^n]_\eta$ and every $u_{\mathsf{q}} \in [\mathsf{U}]_\mu$ there always exists $x'_{\mathsf{q}} \in [\mathbb{R}^n]_\eta$ such that $x_{\mathsf{q}} \xrightarrow[\mathsf{q}]{u_{\mathsf{q}}} x'_{\mathsf{q}}$. This can be seen since by definition of $[\mathbb{R}^n]_\eta$, for any $\widehat{x} \in \mathbb{R}^n$ there always exists a state $\widehat{x}' \in [\mathbb{R}^n]_\eta$ such that $\|\widehat{x} - \widehat{x}'\| \leq \eta$. Hence, for $\overline{\xi}_{x_{\mathsf{q}}u_{\mathsf{q}}}(\tau)$ there always exists a state $x'_{\mathsf{q}} \in [\mathbb{R}^n]_\eta$ satisfying $\left\|\overline{\xi}_{x_{\mathsf{q}}u_{\mathsf{q}}}(\tau) - x'_{\mathsf{q}}\right\| \leq \eta \leq \beta(\theta, \tau) + \gamma(\mu, \tau) + h(\tau, \sigma)\mathsf{e}^{\kappa\tau} + h(\ell\tau, \sigma)\mathsf{e}^{\kappa\ell\tau} + \eta$.

We can now present the main result of the paper showing that any stochastic control system Σ admits a finite symbolic model.

Theorem 4.3: Let $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f, \sigma)$ be any digital stochastic control system. For any $\varepsilon \in \mathbb{R}^+$, and any tuple $\mathsf{q} = (\tau, \eta, \mu, \theta, \ell)$ of quantization parameters satisfying $\mu \leq span(\mathsf{U})$ and $h(\ell\tau, \sigma)\mathsf{e}^{\kappa\ell\tau} + \eta \leq \varepsilon \leq \theta$, we have:

$$S_{\mathsf{q}}(\Sigma) \preceq^{\varepsilon}_{\mathcal{AS}} S_\tau(\Sigma) \preceq^{\varepsilon}_{\mathcal{S}} S_{\mathsf{q}}(\Sigma), \qquad \text{(IV.7)}$$

within the time horizon $0, \tau, \cdots, \ell\tau$.

Before providing the proof, it can be readily seen that when we are interested in the dynamics of Σ on a compact subset $D \subset \mathbb{R}^n$, assumed to be a finite union of boxes, and for a given precision ε, there always exists a small choice of τ such that $h(\tau, \sigma)\mathsf{e}^{\kappa\tau} < \varepsilon$. Then by choosing a sufficiently small value of $\eta \leq span(D)$, the condition of Theorem 4.3 is satisfied.

Proof: We start by proving $S_\tau(\Sigma) \preceq^{\varepsilon}_{\mathcal{S}} S_{\mathsf{q}}(\Sigma)$. Consider the relation $R \subseteq X_\tau \times X_{\mathsf{q}}$ defined by $(x_\tau, x_{\mathsf{q}}) \in R$ if and only if $\mathbb{E}\left[\|H_\tau(x_\tau) - H_{\mathsf{q}}(x_{\mathsf{q}})\|\right] = \mathbb{E}\left[\|x_\tau - x_{\mathsf{q}}\|\right] \leq \varepsilon$. Since

$X_{\tau 0} \subseteq \bigcup_{p\in[\mathbb{R}^n]_\eta} \mathcal{B}_\eta(p)$, for every $x_{\tau 0} \in X_{\tau 0}$ there always exists $x_{\mathsf{q}0} \in X_{\mathsf{q}0}$ such that

$$\mathbb{E}\left[\|x_{\tau 0} - x_{\mathsf{q}0}\|\right] = \|x_{\tau 0} - x_{\mathsf{q}0}\| \leq \eta \leq \varepsilon.$$

Hence, $(x_{\tau 0}, x_{\mathsf{q}0}) \in R$ and condition (i) in Definition 3.2 is satisfied. Now consider any $(x_\tau, x_\mathsf{q}) \in R$. Condition (ii) in Definition 3.2 is satisfied by the definition of R. Let us now show that condition (iii) in Definition 3.2 holds. Consider any $\upsilon_\tau \in U_\tau$. Choose an input $u_\mathsf{q} \in U_\mathsf{q}$ satisfying

$$\|\upsilon_\tau - u_\mathsf{q}\|_\infty = \|\upsilon_\tau(0) - u_\mathsf{q}(0)\| \leq \mu. \tag{IV.8}$$

Note that the existence of such a u_q is guaranteed by the special shape of U, described in Definition 2.1, and by the inequality $\mu \leq span(\mathsf{U})$, which guarantees that $\mathsf{U} \subseteq \bigcup_{p\in[\mathsf{U}]_\mu} \mathcal{B}_\mu(p)$. Consider the transition $x_\tau \xrightarrow[\tau]{\upsilon_\tau} x'_\tau = \xi_{x_\tau \upsilon_\tau}(\tau)$ in $S_\tau(\Sigma)$. Since any stochastic control system Σ is δ-SFC, we have:

$$\begin{aligned}\mathbb{E}\left[\|x'_\tau - \xi_{x_\mathsf{q} u_\mathsf{q}}(\tau)\|\right] &\leq \beta\left(\mathbb{E}\left[\|x_\tau - x_q\|\right], \tau\right) + \gamma(\|\upsilon_\tau - u_\mathsf{q}\|_\infty, \tau) \\ &\leq \beta(\varepsilon, \tau) + \gamma(\mu, \tau). \end{aligned}\tag{IV.9}$$

Since $\mathbb{R}^n \subseteq \bigcup_{p\in[\mathbb{R}^n]_\eta} \mathcal{B}_\eta(p)$, there exists $x'_\mathsf{q} \in X_\mathsf{q}$ such that

$$\mathbb{E}\left[\|x'_\tau - x'_\mathsf{q}\|\right] \leq h(\ell\tau, \sigma)\mathsf{e}^{\kappa\ell\tau} + \eta. \tag{IV.10}$$

Using the inequalities $\varepsilon \leq \theta$, (IV.9), (IV.10), and triangle inequality, we obtain:

$$\begin{aligned}&\left\|\overline{\xi}_{x_\mathsf{q} u_\mathsf{q}}(\tau) - x'_\mathsf{q}\right\| = \mathbb{E}\left[\left\|\overline{\xi}_{x_\mathsf{q} u_\mathsf{q}}(\tau) - x'_\mathsf{q}\right\|\right] \\ &\leq \mathbb{E}\left[\left\|\overline{\xi}_{x_\mathsf{q} u_\mathsf{q}}(\tau) - \xi_{x_\mathsf{q} u_\mathsf{q}}(\tau)\right\|\right] + \mathbb{E}\left[\|\xi_{x_\mathsf{q} u_\mathsf{q}}(\tau) - \xi_{x_\tau \upsilon_\tau}(\tau)\|\right] \\ &\quad + \mathbb{E}\left[\|\xi_{x_\tau \upsilon_\tau}(\tau) - x'_\mathsf{q}\|\right] \\ &\leq h(\tau,\sigma)\mathsf{e}^{\kappa\tau} + \beta(\varepsilon,\tau) + \gamma(\mu,\tau) + h(\ell\tau,\sigma)\mathsf{e}^{\kappa\ell\tau} + \eta \\ &\leq \beta(\theta,\tau) + \gamma(\mu,\tau) + h(\tau,\sigma)\mathsf{e}^{\kappa\tau} + h(\ell\tau,\sigma)\mathsf{e}^{\kappa\ell\tau} + \eta,\end{aligned}$$

which, by the definition of $S_\mathsf{q}(\Sigma)$, implies the existence of $x_\mathsf{q} \xrightarrow[\mathsf{q}]{u_\mathsf{q}} x'_\mathsf{q}$ in $S_\mathsf{q}(\Sigma)$. Therefore, from inequality (IV.10) and since $h(\ell\tau, \sigma)\mathsf{e}^{\kappa\ell\tau} + \eta \leq \varepsilon$, we conclude that $(x'_\tau, x'_\mathsf{q}) \in R$ and condition (iii) in Definition 3.2 holds.

Now we prove $S_\mathsf{q}(\Sigma) \preceq^\varepsilon_{\mathcal{AS}} S_\tau(\Sigma)$. Consider the relation $R \subseteq X_\tau \times X_\mathsf{q}$, defined in the first part of the proof. For every $x_{\mathsf{q}0} \in X_{\mathsf{q}0}$, by choosing $x_{\tau 0} = x_{\mathsf{q}0}$, we have $\|x_{\tau 0} - x_{\mathsf{q}0}\| = 0$ and $(x_{\tau 0}, x_{\mathsf{q}0}) \in R$ and condition (i) in Definition 3.3 is satisfied. Now consider any $(x_\tau, x_\mathsf{q}) \in R$. Condition (ii) in Definition 3.3 is satisfied by the definition of R. Let us now show that condition (iii) in Definition 3.3 holds. Consider any $u_\mathsf{q} \in U_\mathsf{q}$. Choose the input $\upsilon_\tau = u_\mathsf{q}$ and consider the unique solution process $x'_\tau = \xi_{x_\tau \upsilon_\tau}(\tau) \in \mathbf{Post}_{\upsilon_\tau}(x_\tau)$ in $S_\tau(\Sigma)$. Since any stochastic control system Σ is δ-SFC, we have:

$$\mathbb{E}\left[\|x'_\tau - \xi_{x_\mathsf{q} u_\mathsf{q}}(\tau)\|\right] \leq \beta\left(\mathbb{E}\left[\|x_\tau - x_q\|\right], \tau\right) \leq \beta(\varepsilon, \tau). \tag{IV.11}$$

Since $\mathbb{R}^n \subseteq \bigcup_{p\in[\mathbb{R}^n]_\eta} \mathcal{B}_\eta(p)$, there exists $x'_\mathsf{q} \in X_\mathsf{q}$ such that

$$\mathbb{E}\left[\|x'_\tau - x'_\mathsf{q}\|\right] \leq h(\ell\tau, \sigma)\mathsf{e}^{\kappa\ell\tau} + \eta. \tag{IV.12}$$

Using the inequalities $\varepsilon \leq \theta$, (IV.11), and (IV.12), and triangle inequality, we obtain:

$$\begin{aligned}&\left\|\overline{\xi}_{x_\mathsf{q} u_\mathsf{q}}(\tau) - x'_\mathsf{q}\right\| = \mathbb{E}\left[\left\|\overline{\xi}_{x_\mathsf{q} u_\mathsf{q}}(\tau) - x'_\mathsf{q}\right\|\right] \\ &\leq \mathbb{E}\left[\left\|\overline{\xi}_{x_\mathsf{q} u_\mathsf{q}}(\tau) - \xi_{x_\mathsf{q} u_\mathsf{q}}(\tau)\right\|\right] + \mathbb{E}\left[\|\xi_{x_\mathsf{q} u_\mathsf{q}}(\tau) - \xi_{x_\tau \upsilon_\tau}(\tau)\|\right] \\ &\quad + \mathbb{E}\left[\|\xi_{x_\tau \upsilon_\tau}(\tau) - x'_\mathsf{q}\|\right] \\ &\leq h(\tau,\sigma)\mathsf{e}^{\kappa\tau} + \beta(\varepsilon,\tau) + h(\ell\tau,\sigma)\mathsf{e}^{\kappa\ell\tau} + \eta \\ &\leq \beta(\theta,\tau) + \gamma(\mu,\tau) + h(\tau,\sigma)\mathsf{e}^{\kappa\tau} + h(\ell\tau,\sigma)\mathsf{e}^{\kappa\ell\tau} + \eta,\end{aligned}$$

which, by the definition of $S_\mathsf{q}(\Sigma)$, implies the existence of $x_\mathsf{q} \xrightarrow[\mathsf{q}]{u_\mathsf{q}} x'_\mathsf{q}$ in $S_\mathsf{q}(\Sigma)$. Therefore, from inequality (IV.12) and since $h(\ell\tau, \sigma)\mathsf{e}^{\kappa\ell\tau} + \eta \leq \varepsilon$, we conclude that $(x'_\tau, x'_\mathsf{q}) \in R$ and condition (iii) in Definition 3.3 holds. ■

The following remark readily extends the assertion of Theorem 4.3 to be valid over an infinite time horizon, under an assumption on the observation of the diffusion.

Remark 4.4: Suppose the symbolic model is allowed to periodically observe the system $S_\tau(\Sigma)$ after each period $T := \ell\tau$. Then, the assertion of Theorem 4.3 holds over an infinite horizon, since one can update the initial state of the symbolic model up to precision η with respect to the realization of $S_\tau(\Sigma)$ at time $\ell\tau$, and replicate the same strategy periodically. In particular, if the observation period is the same as sampling time, then the lower bound of ε reduces to $h(\tau, \sigma)\mathsf{e}^{\kappa\tau} + \eta$ by setting $\ell = 1$.

Let us highlight that the observation assumption in Remark 4.4 implicitly requires to enlarge the class of admissible inputs to stochastic ones. That is, the input signal synthesized in symbolic model is deterministic within the time horizon $\ell\tau$, but according to the diffusion observation may change from one realization to another.

We note that the results in [25], explained in the following theorem, are fully recovered by the results in Theorem 4.3 if the stochastic control system Σ is not affected by any noise, implying that $h(t, \sigma)$ is identically zero and δ-SFC property becomes δ-FC property. Correspondingly, the definitions of $S_\tau(\Sigma)$ and $S_\mathsf{q}(\Sigma)$ need slight modifications.

Theorem 4.5: [25] Let $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f, 0_{n\times q})$ be a δ-FC digital control system. For any $\varepsilon \in \mathbb{R}^+$, and any quadruple $\mathsf{q} = (\tau, \eta, \mu, \theta, \ell)$ of quantization parameters satisfying $\ell \in \mathbb{N}$, $\mu \leq span(\mathsf{U})$ and $\eta \leq \varepsilon \leq \theta$, we have:

$$S_\mathsf{q}(\Sigma) \preceq^\varepsilon_{\mathcal{AS}} S_\tau(\Sigma) \preceq^\varepsilon_{\mathcal{S}} S_\mathsf{q}(\Sigma). \tag{IV.13}$$

Remark 4.6: Although we assume that the set U is infinite, Theorem 4.3 still holds when the set U is finite, with the following modifications: first, the system Σ is required to satisfy the property (II.2) for $\upsilon = \upsilon'$; second, assume $U_\mathsf{q} = \mathsf{U}$ and set $\gamma(\mu, \tau) = 0$ in the definition of $S_\mathsf{q}(\Sigma)$.

V. Symbolic Control Design for a Jet Engine

We illustrate the results of this paper over the Moore-Greitzer jet engine model in no-stall mode, which is affected by noise and unstable [13]. In this model, the unstable equilibrium (in the absence of noise) is transferred to the origin ($\phi = 0$ and $\psi = 0$) using the following change of coordinates: $\phi = \Phi - 1$, $\psi = \Psi - \Psi_{c0} - 2$, where Φ is the mass flow, Ψ is the pressure rise and Ψ_{c0} is a constant. The resulting model Σ is:

$$\begin{bmatrix} \mathrm{d}\,\phi \\ \mathrm{d}\,\psi \end{bmatrix} = \begin{bmatrix} -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 \\ \frac{1}{\omega^2}(\phi - \upsilon) \end{bmatrix} \mathrm{d}\,t + \begin{bmatrix} 0.1\phi\,\mathrm{d}\,W_t^1 \\ 0.1\psi\,\mathrm{d}\,W_t^2 \end{bmatrix}, \tag{V.1}$$

where ω is a positive constant parameter set to be equal to 1, $\upsilon(t) = \Phi_T(t) - 1$ is the control input and $\Phi_T(t)$ is the mass flow through the throttle. We work on the subset $D = [-2, 2] \times [-2, 2]$ of the state space of Σ. One can readily verify that Σ satisfies the conditions in Definition 2.1 with $L_x = 13$, $L_u = 1$, and $Z = 0.1$, when we are interested in the behaviors of Σ in D.

We show that Σ satisfies (II.8) by finding a suitable matrix P using SOS programming as described in [5]. The constant κ in (II.8) takes the value 1.5 and the resulting matrix is $P = I_2$.

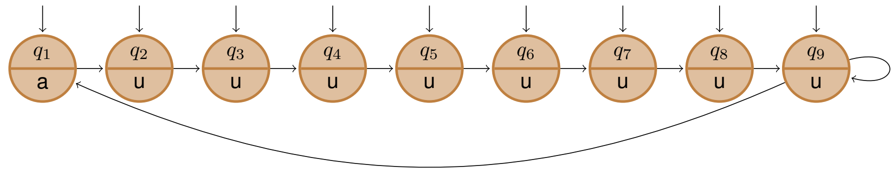

Fig. 1. Finite system describing the schedulability constraint over the controller. The lower part of the states are labeled with the outputs (a and u) denoting the availability and unavailability of the microprocessor, respectively.

Using results of Theorem 2.4, one obtains the following δ-SFC bound for the jet engine model:

$$\mathbb{E}\left[\|\xi_{a\upsilon}(t)-\xi_{a'\upsilon'}(t)\|\right]\leq\sqrt{2}e^{1.5t}\mathbb{E}\left[\|a-a'\|\right]+t\mathsf{e}^{1.5t}|\upsilon-\upsilon'|_{\infty}.$$

For a given sampling time $\tau=0.1$ and choosing a δ-SFC Lyapunov function $V(x,x')=(x-x')^T(x-x')$, one can compute that $h(0.1,\sigma)=0.05$.

We assume that $u\in[-2,2]$ and that the control input can take only three different values from the set $\{-2,0,2\}$. In order to synthesize a controller under this constraint on the input, we select $\mu=2$. The objective is to design a controller forcing the trajectories of the system to reach and stay (in the 1st moment) within the target set $W=[-0.25,0.25]\times[-0.25,0.25]$, which can be expressed in LTL as $\Diamond\Box W$.

Furthermore, we assume that the controller is implemented on a microprocessor, executing other tasks in addition to the control scheme. Let us consider a schedule with epochs of nine time slots, in which at most one slot is allocated to the control task and the rest of them to other tasks. A time slot is an interval of the form $[k\tau,(k+1)\tau[$, with $k\in\mathbb{N}$ and where τ is the sampling time. Therefore, some of the possible microprocessor schedules are given by:

$$|\mathsf{auuuuuuuu}|\mathsf{auuuuuuuu}|\mathsf{auuuuuuuu}|\mathsf{auuuuuuuu}|\cdots,$$

$$|\mathsf{uauuuuuuu}|\mathsf{uauuuuuuu}|\mathsf{uauuuuuuu}|\mathsf{uauuuuuuu}|\cdots,$$

$$|\mathsf{uuuuuuuua}|\mathsf{uuuuuuuuu}|\mathsf{auuuuuuuu}|\mathsf{uuuuuuuuu}|\cdots,$$

where a denotes a slot available for the control task and u denotes a slot allotted to other tasks. The schedulability constraint on the microprocessor can be represented by the finite system (labeled automaton) in Figure 1, where the allowed initial states are distinguished as targets of a sourceless arrow.

Note that we embedded the schedulability constraint by composing the finite system in Figure 1 by the constructed finite system $S_{\mathsf{q}}(\Sigma)$.

For a precision $\varepsilon=0.07$, we construct a symbolic model $S_{\mathsf{q}}(\Sigma)$ by choosing $\theta=0.07$, $\ell=1$, and $\eta=0.01$. The computation of the abstraction $S_{\mathsf{q}}(\Sigma)$ is performed in the tool Pessoa [17] on a laptop with CPU 2 GHz Intel Core i7. The resulting number of states is 1447209. The consumed CPU time for computing the abstraction and synthesizing the controller have been 21554 and 542 seconds, respectively. Here we have assumed that symbolic model can observe the diffusion process at each sampling time, as discussed in Remark 4.4.

Figure 2 displays several realizations of the closed-loop trajectory stemming from the initial conditions in $(-0.75,-1.75)$ and $(1.5,\ 0)$, where the finite system initialized from states q_2 and q_6, respectively. Figure 3 shows the evolution of the input signals corresponding to the above initial conditions.

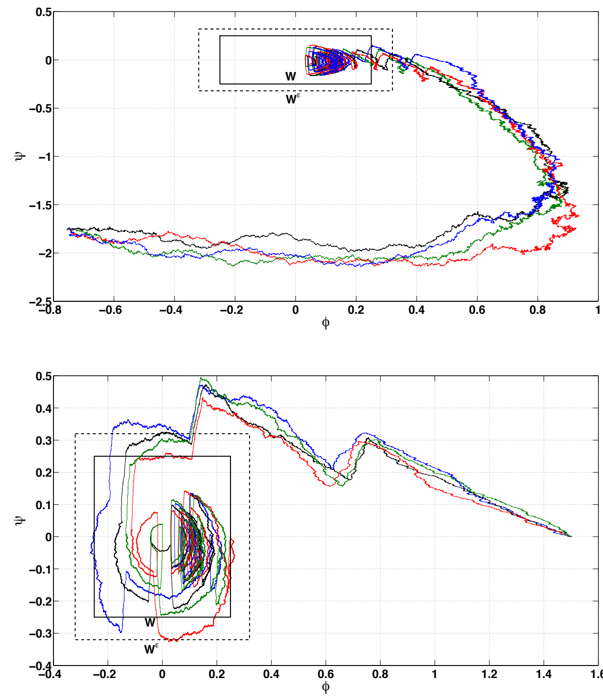

Fig. 2. Several realizations of the closed-loop trajectory with initial conditions $(-0.75,-1.75)$ (upper panel) and $(1.5,\ 0)$ (lower panel), where the finite system initialized from states q_2 and q_6, respectively.

It is readily seen that the specifications are satisfied in the sense that the first moment of the trajectories of Σ reach and stay within $W^{\varepsilon}=[-0.25-\varepsilon,\ 0.25+\varepsilon]\times[-0.25-\varepsilon,\ 0.25+\varepsilon]$, while respecting the schedulability constraint. (The inflation in the set W expressed in the LTL specification by means of the accuravcy parameter ε is intuitive and follows [9].)

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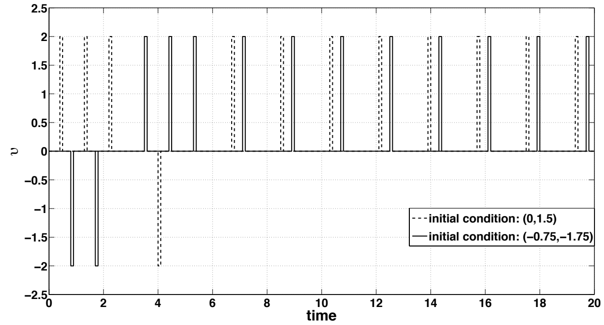

Fig. 3. Evolution of the input signals for two different initial conditions.

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APPENDIX

Proof: [Proof of Lemma 4.1] In the proof, we use the notation $\sigma\sigma^T(x)$ instead of $\sigma(x)\sigma^T(x)$ for the sake of simplicity. In view of Ito's formula and similar to calculation of Lemma 2.5, we have

$$
\begin{aligned}
&\lambda_{\min}(P)\mathbb{E}\left[\left\|\xi_{x\upsilon}(t)-\overline{\xi}_{x\upsilon}(t)\right\|\right]^2 \leq \mathbb{E}\big[V(\xi_{x\upsilon}(t),\overline{\xi}_{x\upsilon}(t))\big]^2\\
&\leq \mathbb{E}\left[V^2(\xi_{x\upsilon}(t),\overline{\xi}_{x\upsilon}(t))\right] =\\
&\int_0^t \mathbb{E}\Big[2\big(\xi_{x\upsilon}(s)-\overline{\xi}_{x\upsilon}(s)\big)^T P\Big(f\big(\xi_{x\upsilon}(s),\upsilon(s)\big)-f\big(\overline{\xi}_{x\upsilon}(s),\upsilon(s)\big)\Big)\\
&\qquad + \operatorname{Tr}\left(\sigma\sigma^T\left(\xi_{x\upsilon}(s)\right)P\right)\Big]ds\\
&=\int_0^t \mathbb{E}\Big[2\big(\xi_{x\upsilon}(s)-\overline{\xi}_{x\upsilon}(s)\big)^T P\Big(f\big(\xi_{x\upsilon}(s),\upsilon(s)\big)-f\big(\overline{\xi}_{x\upsilon}(s),\upsilon(s)\big)\Big)\\
&\qquad + \operatorname{Tr}\left(\sigma\sigma^T\big(\xi_{x\upsilon}(s)-\overline{\xi}_{x\upsilon}(s)\big)P\right)\\
&\qquad + \operatorname{Tr}\left(\sigma\sigma^T\big(\xi_{x\upsilon}(s)\big)P-\sigma\sigma^T\big(\xi_{x\upsilon}(s)-\overline{\xi}_{x\upsilon}(s)\big)P\right)\Big]ds\\
&\leq \int_0^t 2\kappa\mathbb{E}\big[V^2(\xi_{x\upsilon}(s),\overline{\xi}_{x\upsilon}(s))\big]ds \qquad \text{(V.2)}\\
&\quad + \int_0^t \mathbb{E}\Big[\operatorname{Tr}\left(\sigma\sigma^T\big(\xi_{x\upsilon}(s)\big)P-\sigma\sigma^T\big(\xi_{x\upsilon}(s)-\overline{\xi}_{x\upsilon}(s)\big)P\right)\Big]ds\\
&\leq \int_0^t 2\kappa\mathbb{E}\big[V^2(\xi_{x\upsilon}(s),\overline{\xi}_{x\upsilon}(s))\big]ds \qquad \text{(V.3)}\\
&\quad + \int_0^t \mathbb{E}\left[\operatorname{Tr}\left(\sigma\sigma^T\left(\xi_{x\upsilon}(s)\right)P\right)\right]ds \leq \widehat{h}(t,\sigma)\mathsf{e}^{2\kappa t}
\end{aligned}
$$

where the function $\widehat{h}$ can be computed as

$$
\widehat{h}(t,\sigma) := \int_0^t \mathbb{E}\left[\left\|\sqrt{P}\sigma\big(\xi_{x\upsilon}(s)\big)\right\|_F^2\right]ds. \qquad \text{(V.4)}
$$

Inequality (V.2) is a straightforward consequence of (II.7) where $x' := 0$, and (V.3) follows from Gronwall's inequality. Using Lipschitz continuity assumption on σ, we get:

$$
\begin{aligned}
&\int_0^t \mathbb{E}\left[\left\|\sqrt{P}\sigma\big(\xi_{x\upsilon}(s)\big)\right\|_F^2\right]ds\\
&\leq \left\|\sqrt{P}\right\|_F^2 n\min\{n,q\}Z^2\int_0^t \mathbb{E}\left[\|\xi_{x\upsilon}(s)\|^2\right]ds\\
&\leq \left\|\sqrt{P}\right\|_F^2 n\min\{n,q\}Z^2\int_0^t \left(C_1\|x\|^2\mathsf{e}^{Ks}+C_2\|\upsilon\|_\infty^2 s\mathsf{e}^{Ks}\right)ds\\
&= \left\|\sqrt{P}\right\|_F^2 n\min\{n,q\}Z^2 \cdot\\
&\qquad \left(\frac{C_1}{|K|}\left|\mathsf{e}^{Kt}-1\right|\|x\|^2+\frac{C_2}{K^2}\left|t\mathsf{e}^{Kt}-\mathsf{e}^{Kt}+1\right|\|\upsilon\|_\infty^2\right),
\end{aligned}
$$

where the constants C_1, C_2, and K depend on Lipschitz constants L_x, L_u, and Z in Definition 2.1. Note that when we are interested in the dynamics of Σ on a compact subset $D \subset \mathbb{R}^n$, and by defining:

$$
\begin{aligned}
&h(t,\sigma) = \frac{\left\|\sqrt{P}\right\|_F Z\sqrt{n\min\{n,q\}}}{\sqrt{\lambda_{\min}(P)}}\cdot\\
&\sqrt{\left(\frac{C_1}{|K|}\left|\mathsf{e}^{Kt}-1\right|\sup_{x\in D}\{\|x\|^2\}+\frac{C_2}{K^2}\left|t\mathsf{e}^{Kt}-\mathsf{e}^{Kt}+1\right|\sup_{u\in\mathsf{U}}\{\|u\|^2\}\right)},
\end{aligned}
$$

we obtain:

$$
\mathbb{E}\left[\left\|\xi_{x\upsilon}(t)-\overline{\xi}_{x\upsilon}(t)\right\|\right] \leq h(t,\sigma)\mathsf{e}^{\kappa t}. \qquad \text{(V.5)}
$$

It can be easily verified that the proposed function h meets the conditions of Lemma 4.1. ■