# Construction of Approximations of Stochastic Control Systems: A Compositional Approach

Majid Zamani, Matthias Rungger, and Peyman Mohajerin Esfahani

*Abstract*— **In this paper, we provide a compositional framework for the construction of infinite approximations of interconnected stochastic control systems. Our approach is based on a notion of so-called stochastic simulation functions that are associated with interfaces. The stochastic simulation functions are used to quantify the approximation error while the interfaces are used to lift the controllers synthesized for the approximation to the controllers for the original stochastic system. In the first part of the paper, we analyze interconnected stochastic control systems which consist of several stochastic control subsystems. We derive sufficient conditions that facilitate the compositional construction of stochastic simulation functions together with the associated interfaces. Specifically, we show how to construct a stochastic simulation function with the corresponding interface for the interconnected stochastic control system from the simulation functions and interfaces of the individual stochastic control subsystems. In the second part of the paper, we focus on linear stochastic control systems. We extend a methodology, which is known for the non-probabilistic case, to construct infinite approximations of linear stochastic control systems together with their stochastic simulation functions and the corresponding interfaces. Finally, we illustrate the effectiveness of the proposed results on the interconnection of four linear stochastic control subsystems.**

## I. INTRODUCTION

The design of controllers for complex (stochastic) control systems with respect to some complex specifications, e.g. linear temporal logic (LTL) [2], in a reliable and cost effective way is a grand challenge in the study of many safety-critical systems. One promising direction to overcome those complexity issues is the use of simpler (in)finite approximations of the given systems as a substitute in the controller design process. Those approximations allow us to design controllers for the approximations and then refine the controllers to the ones for the concrete complex systems, while providing us with the quantified errors in this detour controller synthesis scheme.

The last decade has witnessed several results on the construction of (in)finite approximations of continuous-time stochastic control systems. The interested reader can consult the recent results in [13, and references therein] on the construction of finite approximations of stochastic control systems. Using those finite approximations, one can leverage the apparatus of finite-state reactive synthesis [9] towards the problem of synthesizing hybrid controllers enforcing complex logical specifications on the original systems. The results in [7] check if an infinite approximation is formally related to a concrete stochastic control system via a notion of so-called stochastic simulation function, however these results do not extend to the construction of approximations and are computationally tractable only for *autonomous* models (i.e., with no inputs). Note that the proposed results in [13] and [7] take a monolithic view of continuous-time stochastic control systems, where the entire system is approximated. This monolithic view interacts badly with the construction of approximations, whose complexity grows (possibly exponentially) in the number of state variables in the model.

In this paper, we provide a compositional framework for the construction of infinite approximations of interconnected stochastic control systems consisting of several stochastic control subsystems. Our framework is based on a new notion of so-called stochastic simulation functions and associated interfaces. Similar to the proposed notions for non-probabilistic control systems [6], the stochastic simulation function in this paper is used to quantify the error between the approximation and the concrete stochastic control system, while the interface is used to lift a controller for the approximation to a controller for the original system.

In the first part of the paper, we present a sufficient small-gain type condition, similar to the one in [4], that facilitates the construction of a stochastic simulation function together with an associated interface between the approximation and the interconnected stochastic system, from the stochastic simulation functions and interfaces of the individual subsystems. In the second part of the paper, we focus on linear stochastic control systems. We extend the approach in [10] on the construction of approximations of linear non-probabilistic control systems together with their corresponding simulation functions and interfaces to linear stochastic control systems.

Similar approaches on the compositional construction of simulation functions based on small-gain type conditions are proposed in [5] and [12]. In [5], the interconnection of two non-probabilistic control subsystems is studied. We generalize that result by considering interconnections of an arbitrary (but finite) number of stochastic control subsystems. General interconnected stochastic systems with an arbitrary number of subsystems are studied in [12] as well. Although the results in [5], [12] assume there exist approximations of original systems and do not provide a way of constructing them, here, we provide constructive means to compute approximations for the case of linear stochastic control systems.

## II. STOCHASTIC CONTROL SYSTEMS

### A. Notation

We denote by $\mathbb{N}$ the set of nonnegative integer numbers and by $\mathbb{R}$ the set of real numbers. We annotate those symbols with subscripts to restrict them in the obvious way, e.g. $\mathbb{R}_{>0}$ denotes the positive real numbers. The symbols $I_n$, $0_n$, and $0_{n\times m}$ denote the identity matrix, zero vector, and zero matrix in $\mathbb{R}^{n\times n}$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times m}$, respectively. For $a,b\in\mathbb{R}$ with $a\leq b$, we denote the closed, open, and half-open intervals in $\mathbb{R}$ by $[a,b]$, $]a,b[$, $[a,b[$, and $]a,b]$, respectively. For $a,b\in\mathbb{N}$ and $a\leq b$, we use $[a;b]$, $]a;b[$, $[a;b[$, and $]a;b]$ to denote the corresponding intervals in $\mathbb{N}$. Given $N\in\mathbb{N}_{\geq1}$, vectors $x_i\in\mathbb{R}^{n_i}$, $n_i\in\mathbb{N}_{\geq1}$ and $i\in[1;N]$, we use $x=[x_1;\ldots;x_N]$ to denote the vector in $\mathbb{R}^n$ with $n=\sum_{i=1}^N n_i$. Similarly, we use $X=[X_1;\ldots;X_N]$ to denote the matrix in $\mathbb{R}^{n\times m}$ with $n=\sum_{i=1}^N n_i$, given $N\in\mathbb{N}_{\geq1}$, matrices $X_i\in\mathbb{R}^{n_i\times m}$, $n_i\in\mathbb{N}_{\geq1}$, and $i\in[1;N]$. Given a vector $x\in\mathbb{R}^n$, we denote by

M. Zamani and M. Rungger are with the Department of Electrical and Computer Engineering, Technische Universität München, D-80290 Munich, Germany. P. Mohajerin Esfahani is with the Automatic Control Laboratory at ETH Zürich and the Risk Analytics and Optimization Chair at ETH Lausanne, Switzerland. Email: zamani@tum.de, matthias.rungger@tum.de, mohajerin@control.ee.ethz.ch.

$\|x\|$ the Euclidean norm of $x$. Given a matrix $P = \{p_{ij}\} \in \mathbb{R}^{n\times n}$, we denote by $\text{Tr}(P) = \sum_{i=1}^{n} p_{ii}$ the trace of $P$.

Given a function $f : \mathbb{R}^n \to \mathbb{R}^m$ and $\bar{x} \in \mathbb{R}^m$, we use $f \equiv \bar{x}$ to denote that $f(x) = \bar{x}$ for all $x \in \mathbb{R}^n$. If $x$ is the zero vector, we simply write $f \equiv 0$. Given a measurable function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, the (essential) supremum of $f$ is denoted by $\|f\|_\infty$; measurability throughout this paper refers to Borel measurability; we recall that $\|f\|_\infty := \text{(ess)sup}\{\|f(t)\|, t \geq 0\}$. A continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; $\gamma$ is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KL}$ if, for each fixed $s$, the map $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to $r$ and, for each fixed nonzero $r$, the map $\beta(r, s)$ is decreasing with respect to $s$ and $\beta(r, s) \to 0$ as $s \to \infty$.

### B. Stochastic control systems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions of completeness and right continuity [8, p. 48]. Let $(W_t)_{t\geq 0}$ be a $\widetilde{p}$-dimensional $\mathbb{F}$-Brownian motion.

*Definition 2.1:* The class of *stochastic control systems* with which we deal in this paper is the tuple $\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \sigma, \mathbb{R}^q, h)$, where

- $\mathbb{R}^n$ is the state space;
- $\mathbb{R}^m$ is the external input space;
- $\mathbb{R}^p$ is the internal input space;
- $\mathcal{U}$ is a subset of the set of all $\mathbb{F}$-progressively measurable processes with values in $\mathbb{R}^m$; see [8, Def. 1.11];
- $\mathcal{W}$ is a subset of the set of all $\mathbb{F}$-progressively measurable processes with values in $\mathbb{R}^p$;
- $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$ is the drift term which is globally Lipschitz continuous: there exist constants $L_x, L_u, L_w \in \mathbb{R}_{>0}$ such that: $\|f(x, u, w) - f(x', u', w')\| \leq L_x\|x-x'\|+L_u\|u-u'\|+L_w\|w-w'\|$ for all $x, x' \in \mathbb{R}^n$, all $u, u' \in \mathbb{R}^m$, and all $w, w' \in \mathbb{R}^p$;
- $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times\widetilde{p}}$ is the diffusion term which is globally Lipschitz continuous;
- $\mathbb{R}^q$ is the output space;
- $h : \mathbb{R}^n \to \mathbb{R}^q$ is the output map.

A stochastic control system $\Sigma$ satisfies

$$\Sigma : \begin{cases} \mathrm{d}\,\xi(t) = & f(\xi(t), \nu(t), \omega(t))\,\mathrm{d}\,t + \sigma(\xi(t))\,\mathrm{d}\,W_t, \\ \zeta(t) = & h(\xi(t)), \end{cases} \tag{II.1}$$

$\mathbb{P}$-almost surely ($\mathbb{P}$-a.s.) for any $\nu \in \mathcal{U}$ and any $\omega \in \mathcal{W}$, where stochastic process $\xi : \Omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is called a *solution process* of $\Sigma$ and stochastic process $\zeta : \Omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}^q$ is called an *output trajectory* of $\Sigma$. We call the tuple $(\xi, \zeta, \nu, \omega)$ a *trajectory* of $\Sigma$, consisting of a solution process $\xi$, an output trajectory $\zeta$, and input trajectories $\nu$ and $\omega$, that satisfies (II.1) $\mathbb{P}$-a.s.. We also write $\xi_{a\nu\omega}(t)$ to denote the value of the solution process at time $t \in \mathbb{R}_{\geq 0}$ under the input trajectories $\nu$ and $\omega$ from initial condition $\xi_{a\nu\omega}(0) = a$ $\mathbb{P}$-a.s., in which $a$ is a random variable that is $\mathcal{F}_0$-measurable. We denote by $\zeta_{a\nu\omega}$ the output trajectory of the solution process $\xi_{a\nu\omega}$. We emphasize that the postulated assumptions on $f$ and $\sigma$ ensure existence, uniqueness, and strong Markov property of the solution processes [3].

## III. Stochastic Simulation Function

Here, we introduce the notion of stochastic simulation function, inspired by the notion of simulation function in [10], for non-probabilistic control systems with internal and external inputs.

*Definition 3.1:* Let $\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \sigma, \mathbb{R}^q, h)$ and $\hat{\Sigma} = (\mathbb{R}^{\hat{n}}, \mathbb{R}^{\hat{m}}, \mathbb{R}^p, \hat{\mathcal{U}}, \mathcal{W}, \hat{f}, \hat{\sigma}, \mathbb{R}^q, \hat{h})$ be two stochastic control systems with the same internal input and output space dimension. Let $V : \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \to \mathbb{R}_{\geq 0}$ be a twice continuously differentiable function and $\nu_{\hat{\nu}} : \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \times \mathbb{R}^{\hat{m}} \times \mathbb{R}^p \to \mathbb{R}^m$ be a (measurable) function which is globally Lipschitz continuous in the first argument. The function $V$ is called a *simulation function of* $\hat{\Sigma}$ *by* $\Sigma$ and $\nu_{\hat{\nu}}$ is called the associated *interface* if for every $x \in \mathbb{R}^n$, $\hat{x} \in \mathbb{R}^{\hat{n}}$, $\hat{u} \in \mathbb{R}^{\hat{m}}$, $w, \hat{w} \in \mathbb{R}^q$, the inequalities:

$$\alpha(\|h(x) - \hat{h}(\hat{x})\|) \leq V(x, \hat{x}), \tag{III.1}$$

and

$$\begin{aligned} \mathcal{L}^{w,\hat{u},\hat{w}}V(x, \hat{x}) &:= \begin{bmatrix}\partial_x V & \partial_{\hat{x}} V\end{bmatrix} \begin{bmatrix} f(x, \nu_{\hat{\nu}}(x, \hat{x}, \hat{u}, \hat{w}), w) \\ \hat{f}(\hat{x}, \hat{u}, \hat{w}) \end{bmatrix} \\ &+ \frac{1}{2}\text{Tr}\left(\begin{bmatrix}\sigma(x) \\ \hat{\sigma}(\hat{x})\end{bmatrix}\begin{bmatrix}\sigma^T(x) & \hat{\sigma}^T(\hat{x})\end{bmatrix}\begin{bmatrix}\partial_{x,x}V & \partial_{x,\hat{x}}V \\ \partial_{\hat{x},x}V & \partial_{\hat{x},\hat{x}}V\end{bmatrix}\right) \\ &\leq -\lambda V(x, \hat{x}) + \rho_{\text{ext}}(\|\hat{u}\|) + \rho_{\text{int}}(\|w - \hat{w}\|), \end{aligned} \tag{III.2}$$

hold for some constant $\lambda \in \mathbb{R}_{>0}$ and some $\mathcal{K}_\infty$ functions $\alpha, \rho_{\text{ext}}, \rho_{\text{int}}$, where $\alpha$ is a convex function and $\rho_{\text{ext}}, \rho_{\text{int}}$ are concave ones.

We say that a stochastic control system $\hat{\Sigma}$ is *approximately alternatingly simulated* by a stochastic control system $\Sigma$ or $\Sigma$ *approximately alternatingly simulates* $\hat{\Sigma}$, denoted by $\hat{\Sigma} \preceq_{\mathcal{AS}} \Sigma$, if there exists a stochastic simulation function of $\hat{\Sigma}$ by $\Sigma$ as in Definition 3.1. Moreover, we call $\hat{\Sigma}$ an *abstraction* of $\Sigma$.

The following theorem shows the importance of the existence of a simulation function by quantifying the error between $\Sigma$ and its abstraction $\hat{\Sigma}$.

*Theorem 3.2:* Let $\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \sigma, \mathbb{R}^q, h)$ and $\hat{\Sigma} = (\mathbb{R}^{\hat{n}}, \mathbb{R}^{\hat{m}}, \mathbb{R}^p, \hat{\mathcal{U}}, \mathcal{W}, \hat{f}, \hat{\sigma}, \mathbb{R}^q, \hat{h})$. Suppose $V$ is a simulation function of $\hat{\Sigma}$ by $\Sigma$ with the associated interface function $\nu_{\hat{\nu}}$. Then, there exist a $\mathcal{KL}$ function $\beta$ and $\mathcal{K}_\infty$ functions $\gamma_{\text{ext}}$, $\gamma_{\text{int}}$ such that for any $\hat{\nu} \in \hat{\mathcal{U}}$, any $\omega, \hat{\omega} \in \mathcal{W}$, and any random variable $a$ and $\hat{a}$ that are $\mathcal{F}_0$-measurable[1], the inequality

$$\begin{aligned} \mathbb{E}[\|\zeta_{a\nu\omega}(t) - \hat{\zeta}_{\hat{a}\hat{\nu}\hat{\omega}}(t)\|] &\leq \beta(\mathbb{E}[V(a, \hat{a})], t) \\ &+ \gamma_{\text{ext}}(\mathbb{E}[||\hat{\nu}||_\infty]) + \gamma_{\text{int}}(\mathbb{E}[||\omega - \hat{\omega}||_\infty]), \end{aligned} \tag{III.3}$$

holds, where $\nu(t) = \nu_{\hat{\nu}}(\xi(t), \hat{\xi}(t), \hat{\nu}(t), \hat{\omega}(t))$.

The proof is similar to the one of Theorem 3.3 in [13] and is omitted here for the sake of brevity.

*Remark 3.3:* The functions $\beta$, $\gamma_{\text{ext}}$, and $\gamma_{\text{int}}$ as

$$\begin{aligned} &\beta(r, t) := \alpha^{-1}\left(3r\mathsf{e}^{-\lambda t}\right),\ \gamma_{\text{ext}}(r) := \alpha^{-1}\left((3/\lambda)\rho_{\text{ext}}(r)\right), \\ &\gamma_{\text{int}}(r) := \alpha^{-1}\left((3/\lambda)\rho_{\text{int}}(r)\right), \end{aligned} \tag{III.4}$$

satisfy inequality (III.3). Note that if $\alpha^{-1}$ satisfies the triangle inequality, i.e., $\alpha^{-1}(a+b) \leq \alpha^{-1}(a)+\alpha^{-1}(b)$, $\forall a, b \in \mathbb{R}_{\geq 0}$, one can substitute the coefficients 3 in the expressions of $\beta$, $\gamma_{\text{ext}}$, and $\gamma_{\text{int}}$ in (III.4) with 1 to get less conservative upper bound in (III.3).

Note that the importance of the result provided in Theorem 3.2 is that one can synthesize a controller for the abstraction $\hat{\Sigma}$, which is potentially easier (e.g., lower dimension) to enforce some complex specification, for example given in LTL. Then by using the interface $\nu_{\hat{\nu}}$, the controller constructed for the abstraction can be *refined* to a controller for the concrete stochastic control system $\Sigma$. The error, introduced in the

[1] Note that $\mathcal{F}_0$ may be the trivial sigma-algebra, i.e., $a$ and $\hat{a}$ are non-probabilistic initial conditions.

design process by taking the detour through the abstraction, is quantified by inequality (III.3).

In the next section, we work with interconnected stochastic control systems *without* internal inputs, resulting from the interconnection of stochastic control subsystems having both internal and external signals. In this case, the interconnected stochastic control systems reduce to the tuple $\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \sigma, \mathbb{R}^q, h)$ and the drift term becomes $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. In this view, the definition of stochastic simulation function for stochastic control systems without internal inputs simplifies as: the interface function becomes $\nu_{\hat{\nu}} : \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \times \mathbb{R}^{\hat{m}} \to \mathbb{R}^m$ and the term $\rho_{\mathrm{int}}(\|w - \hat{w}\|)$ is omitted in (III.2). Similarly, the results in Theorem 3.2 are modified accordingly, i.e., for systems without internal inputs the inequality (III.3) is not quantified over $\omega, \hat{\omega} \in \mathcal{W}$ and, hence, the term $\gamma_{\mathrm{int}}(\mathbb{E}[\|\omega - \hat{\omega}\|_\infty])$ is omitted as well.

## IV. Compositionality Result

In this section, we analyze interconnected stochastic control systems and show how to construct an abstraction of an interconnected stochastic control system together with the corresponding stochastic simulation function and the interface in a compositional fashion. The definition of the interconnected stochastic control system is based on the notion of interconnected systems introduced in [11].

### A. Interconnected stochastic control systems

We consider $N \in \mathbb{N}_{\geq 1}$ stochastic control subsystems

$$\Sigma_i = (\mathbb{R}^{n_i}, \mathbb{R}^{m_i}, \mathbb{R}^{p_i}, \mathcal{U}_i, \mathcal{W}_i, f_i, \sigma_i, \mathbb{R}^{q_i}, h_i), \quad i \in [1; N]$$

with partitioned internal inputs and outputs

$$\begin{aligned} w_i &= \left[w_{i1}; \ldots; w_{i(i-1)}; w_{i(i+1)}; \ldots; w_{iN}\right], \ w_{ij} \in \mathbb{R}^{p_{ij}} \\ y_i &= [y_{i1}; \ldots; y_{iN}], \ y_{ij} \in \mathbb{R}^{q_{ij}} \end{aligned} \tag{IV.1}$$

and output function

$$h_i(x_i) = [h_{i1}(x_i); \ldots; h_{iN}(x_i)]. \tag{IV.2}$$

We interpret the outputs $y_{ii}$ as *external* outputs, whereas the outputs $y_{ij}$ with $i \neq j$ are *internal* outputs which are used to define the interconnected stochastic control systems. In particular, we assume that the dimension of $w_{ij}$ is equal to the dimension of $y_{ji}$, i.e., the following *interconnection constraints* hold:

$$p_{ij} = q_{ji}, \quad \forall i, j \in [1; N], \ i \neq j. \tag{IV.3}$$

If there is no connection between stochastic control subsystem $\Sigma_i$ and $\Sigma_j$, then we assume that the connecting output function is identically zero for all arguments, i.e., $h_{ij} \equiv 0$. We define the *interconnected stochastic control system* as the following.

*Definition 4.1:* Consider $N \in \mathbb{N}_{\geq 1}$ stochastic control subsystems $\Sigma_i = (\mathbb{R}^{n_i}, \mathbb{R}^{m_i}, \mathbb{R}^{p_i}, \mathcal{U}_i, \mathcal{W}_i, f_i, \sigma_i, \mathbb{R}^{q_i}, h_i)$, $i \in [1; N]$, with the input-output configuration given by (IV.1)-(IV.3). The *interconnected stochastic control system* $\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \sigma, \mathbb{R}^q, h)$, denoted by $\mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$, follows by $n = \sum_{i=1}^N n_i$, $m = \sum_{i=1}^N m_i$, $q = \sum_{i=1}^N q_{ii}$, and functions

$$\begin{aligned} f(x, u) &= [f_1(x_1, u_1, w_1); \ldots; f_N(x_N, u_N, w_N)], \\ \sigma(x) &= [\sigma_1(x_1); \ldots; \sigma_N(x_n)], \\ h(x) &= [h_{11}(x_1); \ldots; h_{NN}(x_N)], \end{aligned}$$

where $u = [u_1; \ldots; u_N]$ and $x = [x_1; \ldots; x_N]$ and with the interconnection variables constrained by $w_{ij} = y_{ji}$ for all $i, j \in [1; N], i \neq j$.

The interconnection of two stochastic control subsystems $\Sigma_i$ and $\Sigma_j$ is illustrated in Figure 1.

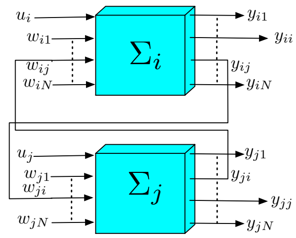


Fig. 1. Interconnection of two stochastic control subsystems $\Sigma_i$ and $\Sigma_j$.

### B. Compositional construction of abstractions, simulation functions, and interfaces

We further assume that we are given $N$ stochastic control subsystems $\Sigma_i = (\mathbb{R}^{n_i}, \mathbb{R}^{m_i}, \mathbb{R}^{p_i}, \mathcal{U}_i, \mathcal{W}_i, f_i, \sigma_i, \mathbb{R}^{q_i}, h_i)$, together with their abstractions $\hat{\Sigma}_i = (\mathbb{R}^{\hat{n}_i}, \mathbb{R}^{\hat{m}_i}, \mathbb{R}^{p_i}, \hat{\mathcal{U}}_i, \mathcal{W}_i, \hat{f}_i, \hat{\sigma}_i, \mathbb{R}^{q_i}, \hat{h}_i)$, with the stochastic simulation functions $V_i$ of $\hat{\Sigma}_i$ by $\Sigma_i$, and with the associated interfaces $\nu_{i\hat{\nu}_i}$. We use $\lambda_i$, $\alpha_i$, $\rho_{i\mathrm{ext}}$, and $\rho_{i\mathrm{int}}$ to denote the corresponding positive constant and $\mathcal{K}_\infty$ functions appearing in Definition 3.1. In order to provide the main compositionality result, we require the following assumption:

*Assumption 1:* For any $i, j \in [1; N]$, $i \neq j$, there exist $\mathcal{K}_\infty$ functions $\gamma_i$ and constants $\hat{\lambda}_i \in \mathbb{R}_{>0}$ and $\delta_{ij} \in \mathbb{R}_{\geq 0}$ such that for any $r \in \mathbb{R}_{\geq 0}$

$$\lambda_i r \geq \hat{\lambda}_i \gamma_i(r) \tag{IV.4a}$$
$$h_{ji} \equiv 0 \implies \delta_{ij} = 0 \text{ and} \tag{IV.4b}$$
$$h_{ji} \not\equiv 0 \implies \rho_{i\mathrm{int}}((N-1)\alpha_j^{-1}(r)) \leq \delta_{ij}\gamma_j(r). \tag{IV.4c}$$

For the ease of notation in the rest of the paper, we define matrices $\Lambda$ and $\Delta$ in $\mathbb{R}^{N \times N}$ with their components given by $\Lambda_{ii} = \hat{\lambda}_i$, $\Delta_{ii} = 0$ for $i \in [1; N]$ and $\Lambda_{ij} = 0$, $\Delta_{ij} = \delta_{ij}$ for $i, j \in [1; N], i \neq j$. Moreover, we define $\Gamma(\hat{s}) := [\gamma_1(s_1); \ldots; \gamma_N(s_N)]$, where $\hat{s} = [s_1; \ldots; s_N]$.

The next theorem provides a compositional approach on the construction of abstractions of interconnected stochastic control systems, of the corresponding stochastic simulation functions, and of the interfaces.

*Theorem 4.2:* Consider the interconnected stochastic control system $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$ induced by $N \in \mathbb{N}_{\geq 1}$ stochastic control subsystems $\Sigma_i$. Suppose that each stochastic control subsystem $\Sigma_i$ approximately alternatingly simulates a stochastic control subsystem $\hat{\Sigma}_i$ with the corresponding stochastic simulation function $V_i$ and interface function $\nu_{i\hat{\nu}_i}$. If Assumption 1 holds and there exists a vector $\mu \in \mathbb{R}^N_{>0}$ such that the inequality

$$\mu^T(-\Lambda + \Delta) < 0 \tag{IV.5}$$

is satisfied[2], then $V(x, \hat{x}) = \sum_i \mu_i V_i(x_i, \hat{x}_i)$ is a stochastic simulation function of $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_1, \ldots, \hat{\Sigma}_N)$ by $\Sigma$ with the following associated interface:

$$\nu_{\hat{\nu}}(x, \hat{x}, \hat{u}) = [\nu_{1\hat{\nu}_1}(x_1, \hat{x}_1, \hat{u}_1, \hat{w}_1); \ldots; \nu_{N\hat{\nu}_N}(x_N, \hat{x}_N, \hat{u}_N, \hat{w}_N)], \tag{IV.6}$$

where

$$\hat{w}_i = \left[\hat{h}_{1i}(\hat{x}_1); \ldots; \hat{h}_{(i-1)i}(\hat{x}_{(i-1)}); \hat{h}_{(i+1)i}(\hat{x}_{(i+1)}); \ldots; \hat{h}_{Ni}(\hat{x}_N)\right].$$

[2]We interpret the inequality component-wise, i.e., for $x \in \mathbb{R}^N$ we have $x < 0$ iff every entry $x_i < 0$, $i \in [1; N]$.

The proof is similar to the one of Theorem 2 in [10] and is omitted due to lack of space.

*Remark 4.3:* As shown in [4, Lemma 3.1], a vector $\mu \in \mathbb{R}^N_{>0}$ satisfying $\mu^T(-\Lambda + \Delta) < 0$ exists if and only if the spectral radius of $\Lambda^{-1}\Delta$ is strictly less than one.

*Remark 4.4:* If the functions $\rho_{i\mathrm{int}}$, $i \in [1; N]$, satisfy the triangle inequality, $\rho_{i\mathrm{int}}(a + b) \leq \rho_{i\mathrm{int}}(a) + \rho_{i\mathrm{int}}(b)$ for all non-negative values of $a$ and $b$, then the condition (IV.4c) reduces to

$$h_{ji} \not\equiv 0 \implies \rho_{i\mathrm{int}}(\alpha_j^{-1}(r)) \leq \delta_{ij}\gamma_j(r). \tag{IV.7}$$

Figure 2 illustrates schematically the result of Theorem 4.2.

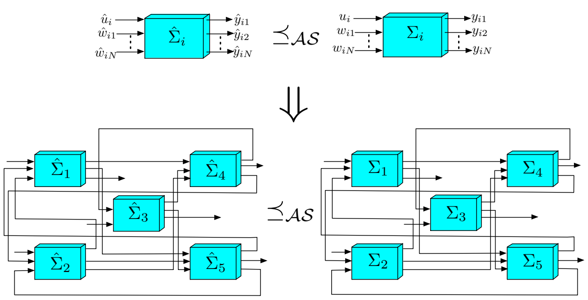

Fig. 2. Compositionality results.

## V. Linear Stochastic Control Systems

In this section, we focus on *linear* stochastic control systems $\Sigma$ and *square-root of quadratic* stochastic simulation functions $V$ with *linear* interfaces $\nu_{\hat{\nu}}$. In the first part, we assume that we are given an abstraction $\hat{\Sigma}$ and provide conditions under which $V$ is a stochastic simulation function. In the second part we show how to construct the abstraction $\hat{\Sigma}$ together with the stochastic simulation function $V$ and corresponding interface $\nu_{\hat{\nu}}$.

### A. Square-root of quadratic stochastic simulation functions

A *linear stochastic control system* is defined as a stochastic control system with the drift, diffusion, and output function given by

$$\begin{aligned} \mathrm{d}\,\xi(t) &= (A\xi(t) + B\nu(t) + D\omega(t))\,\mathrm{d}\,t + E\xi(t)\,\mathrm{d}\,W_t, \\ \zeta(t) &= C\xi(t), \end{aligned} \tag{V.1}$$

where

$$A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, D \in \mathbb{R}^{n\times p}, E \in \mathbb{R}^{n\times n}, C \in \mathbb{R}^{q\times n}.$$

We use the tuple $\Sigma = (A, B, C, D, E, \mathcal{U}, \mathcal{W})$ to refer to a stochastic control system of the form (V.1). Note that in this section we consider linear stochastic control systems driven by a scalar Brownian motion for the sake of simplicity, though the proposed results can be further generalized for the systems driven by multi-dimensional Brownian motions.

In this section, we assume that there exist a constant $\widetilde{\lambda} \in \mathbb{R}_{>0}$ and matrices $M \in \mathbb{R}^{n\times n}$, $K \in \mathbb{R}^{m\times n}$ such that the matrix (in)equalities

$$\begin{aligned} &C^TC \preceq M, \quad M^T = M, \text{ and} \\ &(A + BK)^TM + M(A + BK) \preceq -2\widetilde{\lambda}M, \end{aligned} \tag{V.2}$$

hold. The stabilizability of $(A, B)$ is necessary and sufficient for the existence of such matrices, and one can use various design techniques, e.g., pole placement, in combination with the Lyapunov equation to compute $\widetilde{\lambda}$, $M$, and $K$; see for instance [1] for further details.

Here, we consider square-root of quadratic stochastic simulation functions of the following form

$$V(x, \hat{x}) = \left((x - P\hat{x})^TM(x - P\hat{x})\right)^{\frac{1}{2}} \tag{V.3}$$

with the associated linear interface $\nu_{\hat{\nu}}$ given by

$$\nu_{\hat{\nu}}(x, \hat{x}, \hat{u}, \hat{w}) = K(x - P\hat{x}) + Q\hat{x} + R\hat{u} + S\hat{w} \tag{V.4}$$

where $P$, $Q$, $R$, and $S$ are matrices of appropriate dimensions. Assume that the equalities

$$AP = P\hat{A} - BQ \tag{V.5a}$$
$$D = P\hat{D} - BS \tag{V.5b}$$
$$\hat{C} = CP \tag{V.5c}$$

and the inequality (V.6) hold for some $\lambda \in \mathbb{R}_{>0}$. In the following, we show that those conditions imply that (V.3) is a stochastic simulation function of $\hat{\Sigma}$ by $\Sigma$ with the interface given in (V.4).

*Theorem 5.1:* Consider two linear stochastic control systems $\Sigma = (A, B, C, D, E, \mathcal{U}, \mathcal{W})$ and $\hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{E}, \hat{\mathcal{U}}, \mathcal{W})$ with $p = \hat{p}$ and $q = \hat{q}$. Suppose that there exist constants $\widetilde{\lambda}, \lambda \in \mathbb{R}_{>0}$ and matrices $M, K, P, Q, R, S$ satisfying (V.2), (V.5), and (V.6). Then, $V$ defined in (V.3) is a stochastic simulation function of $\hat{\Sigma}$ by $\Sigma$ with the interface $\nu_{\hat{\nu}}$ given in (V.4).

*Proof:* Note that $V$ is twice continuously differentiable[3] and $\nu_{\hat{\nu}}$ is globally Lipschitz continuous in its first argument. We show that $V$ satisfies $\|Cx - \hat{C}\hat{x}\| \leq V(x, \hat{x})$ and

$$\begin{aligned} \mathcal{L}^{w,\hat{u},\hat{w}}V(x, \hat{x}) :=& \frac{\partial V(x, \hat{x})}{\partial x}(Ax + B\nu_{\hat{\nu}}(x, \hat{x}, \hat{u}, \hat{w}) + Dw) \\ &+ \frac{\partial V(x, \hat{x})}{\partial \hat{x}}(\hat{A}\hat{x} + \hat{B}\hat{u} + \hat{D}\hat{w}) \\ &+ \frac{1}{2}\mathrm{Tr}\left(\begin{bmatrix} Ex \\ \hat{E}\hat{x} \end{bmatrix}\begin{bmatrix} x^TE^T & \hat{x}^T\hat{E}^T \end{bmatrix}\begin{bmatrix} \partial_{x,x}V & \partial_{x,\hat{x}}V \\ \partial_{\hat{x},x}V & \partial_{\hat{x},\hat{x}}V \end{bmatrix}\right) \leq \\ &- \lambda V(x, \hat{x}) + \|\sqrt{M}D\|\|w - \hat{w}\| + \|\sqrt{M}(BR - P\hat{B})\|\|\hat{u}\|, \end{aligned} \tag{V.7}$$

for all $x \in \mathbb{R}^n$, $\hat{x} \in \mathbb{R}^{\hat{n}}$, $\hat{u} \in \mathbb{R}^{\hat{m}}$, $w, \hat{w} \in \mathbb{R}^p$.

From (V.5c), we have $\|Cx - \hat{C}\hat{x}\| = ((x - P\hat{x})^TC^TC(x - P\hat{x}))^{\frac{1}{2}}$ and using $M \succeq C^TC$, it can be readily verified that $\|Cx - \hat{C}\hat{x}\| \leq V(x, \hat{x})$ holds for all $x \in \mathbb{R}^n$, $\hat{x} \in \mathbb{R}^{\hat{n}}$. We proceed with showing the inequality in (V.7). Note that

$$\begin{aligned} &\partial_x V(x, \hat{x}) = \frac{(x - P\hat{x})^TM}{V(x, \hat{x})}, \partial_{\hat{x}}V(x, \hat{x}) = \frac{-(x - P\hat{x})^TMP}{V(x, \hat{x})} \\ &\partial_{x,x}V(x, \hat{x}) = \frac{M}{V(x, \hat{x})} - \frac{M(x - P\hat{x})(x - P\hat{x})^TM}{V^3(x, \hat{x})}, \\ &\partial_{x,\hat{x}}V(x, \hat{x}) = (\partial_{\hat{x},x}V(x, \hat{x}))^T = -\partial_{x,x}V(x, \hat{x})P, \text{ and} \\ &\partial_{\hat{x},\hat{x}}V(x, \hat{x}) = P^T\partial_{x,x}V(x, \hat{x})P \end{aligned}$$

holds. By using the equations (V.5a) and (V.5b) and the definition of the interface function in (V.4) we simplify

$$Ax + B\nu_{\hat{\nu}}(x, \hat{x}, \hat{u}, \hat{w}) + Dw - P(\hat{A}\hat{x} + \hat{B}\hat{u} + \hat{D}\hat{w})$$

to $(A + BK)(x - P\hat{x}) + D(w - \hat{w}) + (BR - P\hat{B})\hat{u}$ and obtain as upper bound for $\mathcal{L}^{w,\hat{u},\hat{w}}V(x, \hat{x})$ as follows:

$$\begin{aligned} &\frac{(x - P\hat{x})^TM\big[(A + BK)(x - P\hat{x}) + D(w - \hat{w}) + (BR - P\hat{B})\hat{u}\big]}{\sqrt{(x - P\hat{x})^TM(x - P\hat{x})}} \\ &+ \frac{\begin{bmatrix} x \\ \hat{x} \end{bmatrix}^T\begin{bmatrix} E^T & 0_{n\times\hat{n}} \\ 0_{\hat{n}\times n} & \hat{E}^T \end{bmatrix}\begin{bmatrix} M & -MP \\ -P^TM & P^TMP \end{bmatrix}\begin{bmatrix} E & 0_{n\times\hat{n}} \\ 0_{\hat{n}\times n} & \hat{E} \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix}}{2\sqrt{(x - P\hat{x})^TM(x - P\hat{x})}}. \end{aligned}$$

[3] Here, we just need $V$ to be twice continuously differentiable over $\mathbb{R}^n \times \mathbb{R}^{\hat{n}} \backslash V_0$, where $V_0 = \{(x, \hat{x}) \in \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \mid V(x, \hat{x}) = 0\}$.

$$-\widetilde{\lambda}\begin{bmatrix} M & -MP \\ -P^TM & P^TMP \end{bmatrix} + \frac{1}{2}\begin{bmatrix} E^T & 0_{n\times\hat{n}} \\ 0_{\hat{n}\times n} & \hat{E}^T \end{bmatrix}\begin{bmatrix} M & -MP \\ -P^TM & P^TMP \end{bmatrix}\begin{bmatrix} E & 0_{n\times\hat{n}} \\ 0_{\hat{n}\times n} & \hat{E} \end{bmatrix} \preceq -\lambda\begin{bmatrix} M & -MP \\ -P^TM & P^TMP \end{bmatrix} \quad \text{(V.6)}$$

We use (V.2) and (V.6) to obtain the following upper bound

$$\frac{(x-P\hat{x})^TM\big[(A+BK)(x-P\hat{x})\big]}{\sqrt{(x-P\hat{x})^TM(x-P\hat{x})}} + \frac{\begin{bmatrix} x \\ \hat{x} \end{bmatrix}^T \begin{bmatrix} E^T & 0_{n\times\hat{n}} \\ 0_{\hat{n}\times n} & \hat{E}^T \end{bmatrix}\begin{bmatrix} M & -MP \\ -P^TM & P^TMP \end{bmatrix}\begin{bmatrix} E & 0_{n\times\hat{n}} \\ 0_{\hat{n}\times n} & \hat{E} \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix}}{2\sqrt{(x-P\hat{x})^TM(x-P\hat{x})}} \leq -\lambda V(x,\hat{x})$$

and with the help of Cauchy-Schwarz inequality to get the following upper bound

$$\frac{(x-P\hat{x})^TM\big[D(w-\hat{w})+(BR-P\hat{B})\hat{u}\big]}{\sqrt{(x-P\hat{x})^TM(x-P\hat{x})}} \leq \|\sqrt{M}D\|\|w-\hat{w}\| + \|\sqrt{M}(BR-P\hat{B})\|\|\hat{u}\|.$$

Using those computed upper bounds, we obtain (V.7) which completes the proof. Note that the $\mathcal{K}_\infty$ functions $\alpha$, $\rho_{\text{ext}}$, and $\rho_{\text{int}}$, in Definition 3.1 associated with the stochastic simulation function in (V.3) are given by $\alpha(s) := s$, $\rho_{\text{ext}}(s) := \|\sqrt{M}(BR-P\hat{B})\|s$ and $\rho_{\text{int}}(s) := \|\sqrt{M}D\|s$, $\forall s \in \mathbb{R}_{\geq 0}$. ■

Note that Theorem 5.1 does not impose any condition on matrix $R$. Similar to the results in [6, Proposition 1] for the non-probabilistic case, we propose a choice of $R$ which minimize function $\rho_{\text{ext}}$. The choice of $R$ minimizing $\rho_{\text{ext}}$ is given by

$$R = (B^TMB)^{-1}B^TMP\hat{B}. \quad \text{(V.8)}$$

As of now, we derived various conditions on the original system $\Sigma$, the abstraction $\hat{\Sigma}$, and the matrices appearing in (V.3) and (V.4), to ensure that (V.3) and (V.4) result in a stochastic simulation function with the associated interface, respectively.

However, those conditions do not impose any requirements on the abstract external input matrix $\hat{B}$. Similar to [6] in the context of non-probabilistic control systems, we choose an external input matrix $\hat{B}$ which *preserves* the behaviors of the original stochastic system $\Sigma$ on the abstraction $\hat{\Sigma}$ in the absence of noise: for every trajectory $(\xi,\zeta,\nu,\omega)$ of $\Sigma$ in the absence of noise there exists a trajectory $(\hat{\xi},\hat{\zeta},\hat{\nu},\hat{\omega})$ of $\hat{\Sigma}$ in the absence of any noise such that $\zeta=\hat{\zeta}$.

Note that using the following choice of external input matrix $\hat{B}$, the results in [10] for the linear control system are fully recovered by the corresponding ones here providing that the linear stochastic control system is not affected by noise, implying that $E$ and $\hat{E}$ are identically zero.

*Theorem 5.2:* Consider two linear stochastic control systems $\Sigma = (A,B,C,D,0_{n\times n},\mathcal{U},\mathcal{W})$ and $\hat{\Sigma} = (\hat{A},\hat{B},\hat{C},\hat{D},0_{\hat{n}\times\hat{n}},\hat{\mathcal{U}},\mathcal{W})$ with $p=\hat{p}$ and $q=\hat{q}$. Suppose that there exist matrices $P$, $Q$, and $S$ satisfying (V.5), and that the abstract external input matrix $\hat{B}$ is given by

$$\hat{B} = [\hat{P}B \;\; \hat{P}AG], \quad \text{(V.9)}$$

where $\hat{P}$ and $G$ are assumed to satisfy

$$C = \hat{C}\hat{P}, \quad I_n = P\hat{P}+GF, \quad \hat{P}P = I_{\hat{n}}, \quad \text{(V.10)}$$

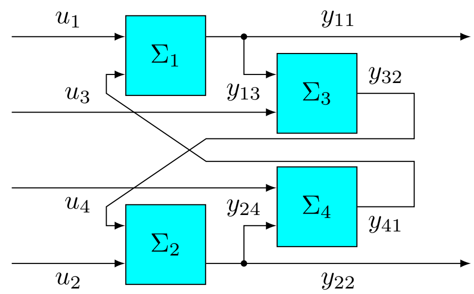


Fig. 3. The interconnected system $\mathcal{I}(\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4)$.

for some matrix $F$. Then, for every trajectory $(\xi,\zeta,\nu,\omega)$ of $\Sigma$ there exists a trajectory $(\hat{\xi},\hat{\zeta},\hat{\nu},\hat{\omega})$ of $\hat{\Sigma}$ so that $\zeta=\hat{\zeta}$ and $\omega=\hat{\omega}$ holds.

The proof is similar to the proof of Theorem 5 in [10] and is omitted here for the lack of space.

In order to compute the abstraction $\hat{\Sigma}$ and the various matrices involved in the definition of the stochastic simulation function and the interface function, one can follow the same chain of lemmas in Subsection 4.3 in [10]. Here, we summarize the construction of the abstraction $\hat{\Sigma}$ in Table I.

| | |
|---|---|
| 1. | Pick an injective $P$ satisfying (26), (27), and (28) in [10]; |
| 2. | Compute $\hat{A}$ and $Q$ from (V.5a); |
| 3. | Compute $\hat{D}$ and $S$ from (V.5b); |
| 4. | Compute $\hat{C}$ from (V.5c); |
| 5. | Compute $\hat{B}$ from (V.9); |
| 6. | Determine $M$, $K$, and $\hat{E}$ so that (V.2) and (V.6) hold. |

TABLE I
CONSTRUCTION OF AN ABSTRACTION $\hat{\Sigma}$.

## VI. AN EXAMPLE

Let us demonstrate the effectiveness of the proposed results on an interconnection of four linear stochastic control subsystems. We consider the system $\mathcal{I}(\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4)$ illustrated in Figure 3, where $\Sigma_i$, $i\in\{1,2\}$, and $\Sigma_j$, $j\in\{3,4\}$, are two triple and two double integrators affected by noise, respectively, with system matrices for $i\in\{1,2\}$ given by

$$A_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -2 & 0 \end{bmatrix},\; B_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\; C_i^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\; E_i = I_3$$

and for $j\in\{3,4\}$ given by

$$A_j = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix},\; B_j = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\; C_j^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\; E_j = I_2.$$

We feed the output of $\Sigma_i$, $i\in\{1,2\}$, to the input of $\Sigma_{i+2}$ and the output of $\Sigma_3$ (resp. $\Sigma_4$) to the input of $\Sigma_2$ (resp. $\Sigma_1$) which we describe with the interconnection matrices that define the output functions $h_{ij}(x_i)=C_{ij}x_i$ by $C_{ii}=C_{i(i+2)}=[1\;\;0\;\;0]$, $i\in\{1,2\}$, and $C_{32}=C_{41}=[1\;\;0]$ and the remaining $h_{ij}\equiv 0$. Correspondingly, the internal input matrices are given by $D_{14}=D_{23}=[0\;\;0\;\;d]^T$ and $D_{(j+2)j}=[0\;\;d]^T$, $d\neq 0$, and $j\in\{1,2\}$. Subsequently, we use $C_i=C_{ii}$, $i\in\{1,2\}$, $C_3=C_{32}$, $C_4=C_{41}$, $D_1=D_{14}$, $D_2=D_{23}$, $D_3=D_{31}$, $D_4=D_{42}$, and denote the stochastic control subsystems by $\Sigma_i=(A_i,B_i,C_i,D_i,E_i,\mathcal{U}_i,\mathcal{W}_i)$.

## A. The abstract subsystems

In order to construct an abstraction for $\mathcal{I}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$ we begin with the construction of the abstractions $\hat{\Sigma}_i$ for each individual subsystem $\Sigma_i$, $i \in \{1,2,3,4\}$. We follow the steps outlined in Table I and obtain from step 1, for $i \in \{1,2\}$ and $j \in \{3,4\}$, the matrices $P_i^T = [1 \quad 0 \quad 0]$ and $P_j^T = [1 \quad 0]$. We continue with steps 2-5 and get the scalar abstract stochastic control subsystems

$$\hat{\Sigma}_{i\in\{1,2,3,4\}} : \begin{cases} \mathrm{d}\,\hat{\xi}_i(t) = \hat{\nu}_i(t)\,\mathrm{d}\,t + \hat{\xi}_i(t)\,\mathrm{d}\,W_t, \\ \hat{\zeta}_i(t) = \hat{\xi}_i(t), \end{cases}$$

with the interconnection matrices $\hat{D}_i = 0$ and diffusion matrices $\hat{E}_i = 1 = \|E_i\|$. Simultaneously, we get $Q_i = -4$ for $i \in \{1,2\}$, $Q_j = -2$ for $j \in \{3,4\}$, and $S_i = -d$ for $i \in \{1,2,3,4\}$. Next, we set $\tilde{\lambda} = 1$ and solve an appropriate linear matrix inequality to determine $M_i$ and $K_i$ so that (V.2) holds. We get

$$M_i = \begin{bmatrix} 6.0122 & 4.3636 & 1.1968 \\ 4.3636 & 4.4916 & 1.2608 \\ 1.1968 & 1.2608 & 0.6304 \end{bmatrix}, \; K_i^T = \begin{bmatrix} -7.5 \\ -3.5 \\ -4.5 \end{bmatrix},$$

for $i \in \{1,2\}$, and

$$M_j = \begin{bmatrix} 6.2601 & 4.6753 \\ 4.6753 & 4.1554 \end{bmatrix}, \; K_j^T = \begin{bmatrix} -4 \\ -3 \end{bmatrix},$$

for $j \in \{3,4\}$. Inequality (V.6) holds for $\lambda = 1/2$. The matrices $R_i$ follow from (V.8) and are $R_i = 1.9$ for $i \in \{1,2\}$ and $R_j = 1.13$ for $j \in \{3,4\}$. The interfaces are given by

$$\begin{aligned} \nu_{i\hat{\nu}_i}(x_i, \hat{x}_i, \hat{u}_i, \hat{w}_i) &= K_i(x_i - P_i\hat{x}_i) - 4\hat{x}_i + 1.9\hat{u}_i - d\hat{w}_i \\ \nu_{j\hat{\nu}_j}(x_j, \hat{x}_j, \hat{u}_j, \hat{w}_j) &= K_j(x_j - P_j\hat{x}_j) - 2\hat{x}_j + 1.13\hat{u}_j - d\hat{w}_j \end{aligned} \tag{VI.1}$$

for $i \in \{1,2\}$ and $j \in \{3,4\}$ and the internal inputs are given by $\hat{w}_1 = \hat{x}_4$, $\hat{w}_2 = \hat{x}_3$, $\hat{w}_3 = \hat{x}_1$, $\hat{w}_4 = \hat{x}_2$. Theorem 5.1 applies to $\Sigma_i$ and $\hat{\Sigma}_i$ showing that $V_i$ of the form (V.3) is a stochastic simulation function of $\hat{\Sigma}_i$ by $\Sigma_i$ with the interface $\nu_{i\hat{\nu}_i}$. As provided in the proof of Theorem 5.1, the comparison functions for $i \in \{1,2\}$ and $j \in \{3,4\}$ are given by

$$\begin{aligned} &\alpha_i(s) = s,\; \lambda_i = \tfrac{1}{2},\; \rho_{i\mathrm{ext}}(s) = 0.65s,\; \rho_{i\mathrm{int}}(s) = 1.76ds, \\ &\alpha_j(s) = s,\; \lambda_j = \tfrac{1}{2},\; \rho_{j\mathrm{ext}}(s) = 0.15s,\; \rho_{j\mathrm{int}}(s) = 3ds, \end{aligned}$$

for any $s \in \mathbb{R}_{\geq 0}$.

## B. The interconnected system

We now proceed by applying Theorem 4.2. In particular, we check Assumption 1, which is satisfied by $\gamma_i(s) = s$, $\hat{\lambda}_i = \frac{1}{2}$, and using $\rho_{i\mathrm{int}}$ (c.f. Remark 4.4), $\delta_{ij}$ are as $\delta_{13} = \delta_{24} = 1.76d$, $\delta_{32} = \delta_{41} = 3d$, and the rests are zero. Additionally, we require the existence of a vector $\mu \in \mathbb{R}^4_{>0}$ satisfying (IV.5), which is the case if and only if the spectral radius of $2\Delta$ is strictly less than one, i.e., $2\sqrt{3 \times 1.76d} < 1$, which holds for $d = 0.1$. One can choose the vector $\mu$ as $\mu = [1; 1; 1; 1]$ and, hence, it follows that $V(x, \hat{x}) = \sum_{i=1}^4 V_i(x_i, \hat{x}_i)$ is a stochastic simulation function of $\mathcal{I}(\hat{\Sigma}_1, \hat{\Sigma}_2, \hat{\Sigma}_3, \hat{\Sigma}_4)$ by $\mathcal{I}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$ where the interface function follows from (VI.1). Following the proof of [10, Theorem 2], we see that $V$ satisfies (III.1) with $\alpha(s) = s$ and (III.2) with $\lambda = 0.4$, $\rho_{\mathrm{ext}}(s) = 0.65s$, and $\rho_{\mathrm{int}} \equiv 0$. From Theorem 3.2 we obtain

$$\mathbb{E}[\|\zeta_{a\nu\omega}(t) - \hat{\zeta}_{\hat{a}\hat{\nu}\hat{\omega}}(t)\|] \leq \mathrm{e}^{-0.4t}\mathbb{E}[V(a, \hat{a})] + 4.6\mathbb{E}[\|\hat{\nu}\|_\infty]. \tag{VI.2}$$

We show some simulation results in Figure 4 for inputs

$$\hat{\nu}_1(t) = \tfrac{1}{1+t}\sin(t),\; \hat{\nu}_2(t) = \tfrac{1}{1+t}\cos(t),\; \hat{\nu}_4 = \hat{\nu}_3 \equiv 0.$$

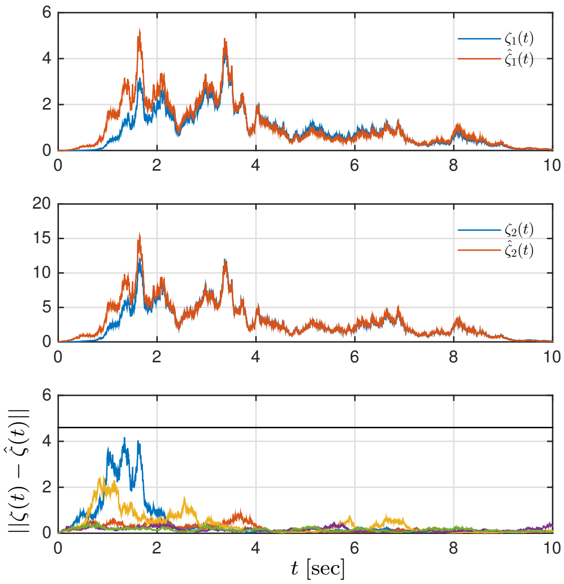


Fig. 4. Top two plots: One realization of $\zeta_1$ (resp. $\zeta_2$) (—) and $\hat{\zeta}_1$ (resp. $\hat{\zeta}_2$) (—). Bottom: Five realizations of $\|\zeta - \hat{\zeta}\|$. The solid black line indicates the error bound 4.6 as computed using (VI.2).

In the top two plots of the figure, we see a realization of the observed process, $\zeta_1$ (resp. $\zeta_2$) and $\hat{\zeta}_1$ (resp. $\hat{\zeta}_2$) of $\mathcal{I}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$ and $\mathcal{I}(\hat{\Sigma}_1, \hat{\Sigma}_2, \hat{\Sigma}_3, \hat{\Sigma}_4)$, respectively. On the bottom part, we see five realizations of $\|\zeta - \hat{\zeta}\|$, where $\zeta = [\zeta_1; \zeta_2]$ and $\hat{\zeta} = [\hat{\zeta}_1; \hat{\zeta}_2]$. The solid black line denotes the error bound given by the computed stochastic simulation function $V$ (see Theorem 3.2).

## References


[1] P. J. Antsaklis and A. N. Michel. *A Linear Systems Primer*. Springer, 2007.

[2] C. Baier and J. P. Katoen. *Principles of model checking*. The MIT Press, April 2008.

[3] V.S. Borkar. *Optimal control of diffusion processes*. Pitman research notes in mathematics series. Longman Scientific & Technical, 1989.

[4] S. Dashkovskiy, H. Ito, and F. Wirth. On a small gain theorem for iss networks in dissipative lyapunov form. *European Journal of Control*, 17(4):357–365, 2011.

[5] A. Girard. A composition theorem for bisimulation functions, 2007. Preprint.

[6] A. Girard and G. J. Pappas. Hierarchical control system design using approximate simulation. *Automatica*, 45(2):566–571, 2009.

[7] A. A. Julius and G. J. Pappas. Approximations of stochastic hybrid systems. *IEEE Transaction on Automatic Control*, 54(6):1193–1203, 2009.

[8] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2nd edition, 1991.

[9] O. Maler, A. Pnueli, and J. Sifakis. On the synthesis of discrete controllers for timed systems. In E. W. Mayr and C. Puech, editors, *Symposium on Theoretical Aspects of Computer Science*, volume 900 of *LNCS*, pages 229–242. Springer-Verlag, 1995.

[10] M. Rungger and M. Zamani. Compositional construction of approximate abstractions. In *Proceedings of the 18th International Conference on Hybrid Systems: Computation and Control*. ACM New York, NY, USA, April 2015.

[11] Y. Tazaki and J. Imura. Bisimilar finite abstractions of interconnected systems. In *Proc. of the 11th Int. Conf. on Hybrid Systems: Computation and Control*, pages 514–527. Springer, 2008.

[12] M. Zamani. Compositional approximations of interconnected stochastic hybrid systems. In *Proceedings of the 53rd IEEE Conference on Decision and Control*, pages 3395–3400, December 2014.

[13] M. Zamani, P. Mohajerin Esfahani, R. Majumdar, A. Abate, and J. Lygeros. Symbolic control of stochastic systems via approximately bisimilar finite abstractions. *IEEE Transactions on Automatic Control, Special Issue on Control of Cyber-Physical Systems*, 59(12):3135–3150, December 2014.