A Linear Programming Approach to Design Online Triggering Mechanisms for Robust MPC

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***Abstract*— In this paper, an event-triggering approach is proposed for a robust model predictive control method. The approach is applicable to constrained, linear time-invariant systems with bounded, additive disturbances. At each triggering instant, the triggering mechanism is designed online using a linear programming approach. Intuitively, the mechanism is a sequence of hyper-rectangles that surround the optimal state trajectory, over the prediction horizon. Standard analyses of robust feasibility and robust stability of the closed-loop, event-triggered control system are conducted. A numerical example is presented to show benefits of the proposed approach. In particular and under the assumption that the disturbance has a uniform distribution, we further study some statistical properties of the generated triggering instants.**

I. INTRODUCTION

Controlled physical systems are generally continuous-time phenomena which are operated on or manipulated via communication and computation components. These components usually have discrete nature (whether in a temporal or a spatial form), e.g., a wireless communication unit or a digital sensor. As a result, the implementation of designed control laws is affected by the discrete character of computation and communication components. We refer to such systems as *cyber-physical* systems [17]. A common approach to implement control laws under such circumstances is the so-called sample-and-hold approach. A traditional practice in the sample-and-hold approach is to periodically update the control action [3]. It has been shown that one can guarantee desired stability and/or performance measures under suitable conditions.

There is a class of cyber-physical systems that is called wireless networked control systems (WNCSs). By the definition, WNCSs are control systems in which feedback loops are closed via wireless communication networks [4]. These systems possess nice properties such as the flexibility in design and the ease of maintenance. However, these systems have some fundamental limitations such as the limited bandwidth and battery life.

Åström and Bernhardsson in [2] have reported several potential benefits of closing a feedback loop in sampled-data systems in a different way. They called this new approach the "Lebesgue sampling" approach. In particular, they used the deviation law $\|x(t) - x(t_k)\| \leq \delta$ to determine the next sampling time t_{k+1}, where the parameter $t \in [t_k, t_{k+1})$ denotes the time, $x(t)$ is the current state, $x(t_k)$ denotes the last sampled state, $\{t_k\}_{k\geq 0}$ is the sequence of generated sampling instants, and finally the positive constant δ is a user-defined parameter. They also pointed out the possibility of reducing the required communication and control law computation frequencies via this new type of sampling. It is natural to expect that the aforementioned benefits of this state-dependent sampling approach may alleviate some shortcomings of WNCSs. Later on and in a rigorous fashion, Tabuada in [20] proposed a state-dependent sampling law $\|x(t) - x(t_k)\| \leq \sigma \|x(t_k)\|$ for input-to-state stable control systems, where σ is a suitable positive constant. In the literature, these types of state-dependent sampling laws are known as *event-triggering mechanism* and the corresponding control systems are called *event-triggered control* (ETC) systems. We refer the interested reader to [11] and [14] and the references therein for reviews of ETC systems.

A particular class of control methods that suits such an event-triggering setup is the class of robust model predictive control (RMPC) methods [8]. The reasons behind such a compatibility can be at least twofold. First, RMPC methods are online, optimization-based methods and are computationally expensive. As a result, there is hope to reduce the frequency with which the underlying optimization problem is solved. Second, there exists a pair of optimal input and state trajectories as the outcome of the optimization problem at each sampling instant. In a standard implementation of RMPC only the first element in the optimal input trajectory is used [15]. It is hence logical to exploit these optimal trajectories and to amend RMPC methods with event-triggering implementations.

There are numerous studies in the literature that have pursued such an amendment to RMPC methods. The authors in [10] propose an online, event-triggering RMPC approach. Their approach is capable of identifying the necessary input-channel to be updated. A Lebesgue sampling approach is used to design an estimator with a bounded covariance matrix in [19]. The authors in [7] use the notion of *explicit* MPC [6] to construct an ℓ_1-type triggering mechanism. Based on a user-defined 2-norm ball around the optimal state trajectory, a Lebesgue-type triggering mechanism is proposed in [13] for WNCSs. Under some mild regularity assumptions, the authors in [1] introduce a co-design event-triggering MPC approach that outperforms a standard MPC method (in the sense of closed-loop performance/average transmission rate). Considering the number of control messages is limited, an event-triggering mechanism is proposed in [9].

In this paper, we consider linear time-invariant (LTI) systems with bounded additive disturbance. We propose a

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Lebesgue-type triggering mechanism for the RMPC method introduced in [18]. The triggering mechanism is a sequence of hyper-rectangles constructed around the optimal state trajectory (that is the solution of the RMPC problem solved at a triggering instant). Furthermore, the triggering mechanism is designed online using a linear programming approach. Since the construction approach is online, it is computationally more expensive with respect to the user-defined Lebesgue triggering methods in the literature. On a positive note, since the system dynamics and the structure of input/state constraints actively determine the derived triggering mechanism, the overall convergence and feasibility properties of the closed-loop dynamics do not get affected as a result of the event-triggered implementation. We also show that when the closed-loop system reaches the desired target sets, some statistical properties of the generated triggering instants are superior to the standard implementation of the RMPC method. We finally note that an extended version of this work is available in [12]. In particular, the interested reader is referred to [12] for the proofs of theoretical results and an in-depth discussion about the computational aspects of our proposed approach.

Notation: the set of non-negative integers is denoted by $\mathbb{Z}_{\geq 0}$. Given positive integers m and n, $\mathbb{R}^m$ and $\mathbb{R}^{m\times n}$ represent the m-dimensional Euclidean space and the space of $m \times n$ matrices with real entries, respectively. Given a positive integer r, the sets of positive integers and non-negative integers less than or equal to r are denoted by $\mathsf{N}_{[r]}$ and $\mathsf{Z}_{[r]}$, respectively. Given a vector $v \in \mathbb{R}^n$, v^i represents the i-th entry of v. For any pairs of vectors $a, b \in \mathbb{R}^n$, the inequality $a < (\leq) b$ is realized in a component-wise manner, i.e., $a^i < (\leq) b^i$, for all $i \in \mathsf{N}_{[n]}$. Given a matrix $M \in \mathbb{R}^{m\times n}$, M_{ij} denotes the i-th row, j-th column entry of M. Moreover, the matrix $M^+ \in \mathbb{R}^{m\times n}$ is the matrix with entries of $M^+_{ij} := \max\{0, M_{ij}\}$. A positive definite matrix M is denoted by $M \succ 0$. The $n \times n$ identity matrix is denoted by I_n. Given a vector $v \in \mathbb{R}^n$ and a scalar $p \geq 1$, $\|v\|_p$ denotes the p-norm $\left(\sum_{i=1}^n \ (v^i)^p\right)^{1/p}$. The function $\text{sign}(\cdot)$ represents the standard sign function. Given a set $\mathcal{S} \subset \mathbb{R}^n$ and a matrix $M \in \mathbb{R}^{m\times n}$, the set $M\mathcal{S}$ denotes the set $\{c \in \mathbb{R}^m : \text{ there exist } s \in \mathcal{S} \text{ such that } Ms = c\}$. Given two sets $\mathcal{A}$ and $\mathcal{B}$ in $\mathbb{R}^n$, $\mathcal{A}/\mathcal{B} := \{x \in \mathcal{A} : \ x \notin \mathcal{B}\}$. Given a matrix $M \succ 0$, the squared weighted distance of a point $r \in \mathbb{R}^n$ from a set $\mathcal{S} \subset \mathbb{R}^n$ is defined as
$d(r, \mathcal{S}, M) := \min_{s\in\mathcal{S}} ||r-s||^2_M = \min_{s\in\mathcal{S}}(r-s)^\top M(r-s)$. Given two sets $\mathcal{C}$ and $\mathcal{D}$, $\mathcal{C} \sim \mathcal{D}$ and $\mathcal{C} \oplus \mathcal{D}$ denote the Pontryagin difference and the Minkowski sum of these sets, respectively. A set $\mathcal{S} \subset \mathbb{R}^n$ is called a polyhedron, if

$$\mathcal{S} := \{s \in \mathbb{R}^n : A_{\mathcal{S}} s \leq b_{\mathcal{S}}\}, \ A_{\mathcal{S}} \in \mathbb{R}^{m\times n}, \ b_{\mathcal{S}} \in \mathbb{R}^m.$$

If in addition the polyhedron $\mathcal{S}$ is bounded, the set is also called a *polytope* and the representation is known as the *H-representation*. For any vector-pairs $l, u \in \mathbb{R}^n$ such that $l < u$, the full-dimensional convex polytope $\mathcal{B}(l, u) := \{x \in \mathbb{R}^n : \ l \leq x \leq u\} = \{x \in \mathbb{R}^n : \ A_{\mathcal{B}} x \leq b_{\mathcal{B}}\}$ is called a *hyper-rectangle*, where $A_{\mathcal{B}} := [\mathsf{I}_n \ \ -\mathsf{I}_n]^\top$ and $b_{\mathcal{B}} = [u^\top \ \ -l^\top]^\top$.

II. Problem Formulation

In this section, we first introduce the class of constrained dynamical systems considered in this paper. The description of the RMPC method is then presented. At last, we formally state the problem addressed in this paper.

A. System Description

Consider a controllable LTI system with bounded additive perturbations given by

$$x_{k+1} = Ax_k + Bu_k + w_k, \ \forall k \in \mathbb{Z}_{\geq 0}, \tag{1}$$

where x_k, u_k, and w_k denote the state, input, and disturbance, respectively. The state, input and disturbance signals satisfy the constraints

$$x_k \in \mathbb{X} \subset \mathbb{R}^{n_x}, \ u_k \in \mathbb{U} \subset \mathbb{R}^{n_u}, \ w_k \in \mathcal{W} \subset \mathbb{R}^{n_x}. \tag{2}$$

The sets $\mathbb{X}$, $\mathbb{U}$, and $\mathcal{W}$ are all convex polytopes and contain the origin of their corresponding space. The nominal system associated with (1) is

$$\bar{x}_{k+1} = A\bar{x}_k + Bu_k, \ \forall k \in \mathbb{Z}_{\geq 0}. \tag{3}$$

We now introduce the RMPC method proposed in [18]. The goal in this method is to guarantee that x_k and u_k converge to some *target sets* $\mathbb{T}_x \subset \mathbb{X}$ and $\mathbb{T}_u \subset \mathbb{U}$ as $k \to \infty$, respectively. These target sets are both user-defined, convex polytopes and contain the origin of their space, as well. The optimization problem $\mathcal{P}(x_k)$ for a finite horizon N at the instant k is

$$\mathcal{P}(x_k) : J(x_k, \mathbf{U}^*_{k|k}) = \min_{\mathbf{X}_{k|k}, \mathbf{U}_{k|k}} J(x_k, \mathbf{U}_{k|k}) \tag{4a}$$

subject to

$$\bar{x}_k = x_k, \tag{4b}$$

$$\bar{x}_{k+i+1|k} = A\bar{x}_{k+i|k} + Bu_{k+i|k}, \ \forall i \in \mathsf{Z}_{[N-1]}, \tag{4c}$$

$$u_{k+i|k} \in \mathcal{U}_i, \ \forall i \in \mathsf{Z}_{[N-1]}, \tag{4d}$$

$$\bar{x}_{k+i|k} \in \mathcal{X}_i, \ \forall i \in \mathsf{Z}_{[N-1]}, \tag{4e}$$

$$\bar{x}_{k+N|k} \in \mathcal{X}_f, \tag{4f}$$

where $\mathbf{U}^*_{k|k} := \{u^*_{k+i|k}\}_{i=0}^{N-1}$ and $\mathbf{X}^*_{k|k} := \{x^*_{k+i|k}\}_{i=0}^{N}$ denote the optimal input and state trajectories of $\mathcal{P}(x_k)$, respectively. The cost function of the RMPC problem is

$$J(x_k, \mathbf{U}_{k|k}) = \sum_{i=0}^{N-1} d(\bar{x}_{k+i|k}, \mathcal{T}_{x,i}, Q) + d(u_{k+i|k}, \mathcal{T}_{u,i}, R), \tag{5}$$

where d is the weighted distance function defined in Notation section. For all $i \in \mathsf{Z}_{[N-2]}$, the input, state, input target, and state target sets are tightened as follows:

$$\mathcal{U}_0 = \mathbb{U}, \qquad \mathcal{U}_{i+1} = \mathcal{U}_i \sim K_i L_i \mathcal{W}, \tag{6a}$$

$$\mathcal{X}_0 = \mathbb{X}, \qquad \mathcal{X}_{i+1} = \mathcal{X}_i \sim L_i \mathcal{W}, \tag{6b}$$

$$\mathcal{T}_{u,0} = \mathbb{T}_u, \quad \mathcal{T}_{u,i+1} = \mathcal{T}_{u,i} \sim K_i L_i \mathcal{W}, \tag{6c}$$

$$\mathcal{T}_{x,0} = \mathbb{T}_x, \quad \mathcal{T}_{x,i+1} = \mathcal{T}_{x,i} \sim L_i \mathcal{W}, \tag{6d}$$

where

$$L_0 = \mathsf{I}_{n_x}, \quad L_{i+1} = (A + BK_i)L_i. \tag{6e}$$

Moreover, the set of gains $\mathbf{K} = \{K_i\}_{i=0}^{N-1}$ are some M-step nilpotent gains where the positive constant M satisfies $n_x \leq M \leq N - 1$. Since the nominal system is controllable, one can for example use the procedure introduced in [18] to construct the set $\mathbf{K}$. On the other hand, the controllability of the nominal system (3) assures that the corresponding discrete algebraic Riccati equation [16] has a unique solution $P \succ 0$, for some positive definite matrices Q and R. One can then employ P to find a stabilizing gain $F = -(R+B^\top PB)^{-1}B^\top A$. Next, suppose $\mathcal{R}$ is a robust control invariant set under the (linearly mapped) disturbance set $L_{N-1}\mathcal{W}$ such that if $x \in \mathcal{R}$,

$$(A + BF)x + L_{N-1}w \in \mathcal{R},\ \forall w \in \mathcal{W}, \tag{7a}$$
$$x \in \mathcal{X}_{N-1}, \tag{7b}$$
$$x \in \mathcal{T}_{x,N-1}, \tag{7c}$$
$$Fx \in \mathcal{U}_{N-1}, \tag{7d}$$
$$Fx \in \mathcal{T}_{u,N-1}. \tag{7e}$$

We then define the (non-empty) terminal state set $\mathcal{X}_f$ by

$$\mathcal{X}_f = \mathcal{R} \sim L_{N-1}\mathcal{W} \subset \mathbb{R}^{n_x}. \tag{8}$$

B. Problem Setup

We are now set to the introduce the problem addressed in this paper. Consider the dynamical system (1) subject to the constraints (2). Let $k \in \mathbb{Z}_{\geq 0}$ be the last instant at which the problem (4) is solved with the corresponding optimal trajectories $\mathbf{U}^*_{k|k}$ and $\mathbf{X}^*_{k|k}$. Finally, assume the controller transmits the optimal input trajectory $\mathbf{U}^*_{k|k}$ to the actuators.

Problem 2.1: Construct a triggering mechanism in the form of a sequence of hyper-rectangles $\mathcal{E}_k := \{\mathcal{E}_{j,k}\}_{j=1}^{N-1}$ to determine the next triggering instant

$$k_{\text{trig}} := k + \min\{j \in \mathsf{N}_{[N-1]} :\ x_{x+j} \notin x^*_{k+j|k} \oplus \mathcal{E}_{j,k}\}, \tag{9}$$

such that (i) the dynamics (1) respect the constraints (2), and (ii) $x_k \to \mathbb{T}_x$ and $u_k \to \mathbb{T}_u$, as $k \to \infty$, if

- the actuator units employ the control action $u_{k+j} = u^*_{k+j|k}$, for all $j \in \mathsf{Z}_{[k_{\text{trig}}-k]}$.

Notice that x_{k+j} is the observed state at the sensors and the controller will transmit $\{x_{k+j|k} \oplus \mathcal{E}_{j,k}\}_{j=1}^{N-1}$ to the sensors.

III. Main Results

In this section, we begin with explaining how each hyper-rectangle $\mathcal{E}_{j,k}$ is constructed. We then present the results concerning the robust feasibility and the robust convergence of the corresponding event-triggered RMPC method.

A. Construction of $\mathcal{E}_{j,k}$

The procedure to construct each set $\mathcal{E}_{j,k}$ is now explained. Denote the last triggering instant by k with the corresponding optimal trajectories $\mathbf{U}^*_{k|k}$ and $\mathbf{X}^*_{k|k}$. Moreover, let $j \in \mathsf{N}_{[N-1]}$ be a time instant following k (indicating the instant that the triggering mechanism is being evaluated). Suppose next the sensors observe the state vector x_{k+j} at the instant $k+j$. Define the prediction error

$$e_{k+j|k} = x_{k+j} - x^*_{k+j|k}. \tag{10}$$

Since $\mathcal{E}_{j,k}$ is a hyper-rectangle, this set can be represented as $\mathcal{E}_{j,k} := \left\{\epsilon \in \mathbb{R}^{n_x} :\ -\underline{e}^p_{j,k} \leq \epsilon^p \leq \bar{e}^p_{j,k},\ \forall p \in \mathsf{N}_{[n_x]}\right\}$, where ϵ^p is the p-th element of ϵ. Each hyper-rectangle $\mathcal{E}_{j,k}$ is parameterized in $2n_x$ parameters $\underline{e}^p_{j,k}$ and $\bar{e}^p_{j,k}$. In light of the definition (10), the triggering mechanism (9) can now be rewritten as $k_{\text{trig}} := k + \min\{j \in \mathsf{N}_{[N-1]} :\ e_{k+j|k} \notin \mathcal{E}_{j,k}\}$.

Our aim is now to find the required conditions on the triggering set $\mathcal{E}_{j,k}$ by which the feasibility and stability of the event-based implementation are ensured. For each possible prediction error $e_{k+j} \in \mathcal{E}_{j,k}$, there should exist a corresponding pair of *candidate* input and state trajectories. These candidate trajectories in turn should guarantee the recursive feasibility and the robust stability of the closed-loop dynamics. In doing so, the first step is to properly define a certain type of optimal input and state trajectories at the instant $k+j$ with the horizon N. (Notice that we have access to $\mathbf{U}^*_{k|k}$ and $\mathbf{X}^*_{k|k}$ from solving $\mathcal{P}(x_k)$.) To this end, we propose

$$u^*_{k+j+i|k} = FA_{\text{cl}}^{j+i-N}x^*_{k+N|k}, \qquad \text{if } j+i \geq N, \tag{11a}$$
$$x^*_{k+j+i|k} = A_{\text{cl}}^{j+i-N}x^*_{k+N|k}, \qquad \text{if } j+i \geq N+1, \tag{11b}$$

for the missing entries of the optimal trajectories, where $A_{\text{cl}} := A + BF$.

Next, we try to enforce a set of conditions on the triggering set $\mathcal{E}_{j,k}$ such that the inter-event feasibility and the recursive feasibility are guaranteed. These conditions ensure that if $e_{j+k} \in \mathcal{E}_{j,k}$, one can construct a pair of candidate input and state trajectories that satisfy the constraints (4b)-(4f). Moreover, these trajectories ensure that the problem (4) remains feasible at triggering instants. A possible way to capture these properties is provided below. Let us first define the gains $\tilde{K}_i$ and the state-transition matrices $\tilde{L}_i$,

$$\tilde{K}_0 := 0_{n_u \times n_x},\ \tilde{K}_{i+1} := K_i,\ \forall i \in \mathsf{Z}_{[N-2]}, \tag{12a}$$
$$\tilde{L}_0 := \mathsf{I}_{n_x},\ \tilde{L}_{i+1} := (A + B\tilde{K}_i)\tilde{L}_i,\ \forall i \in \mathsf{Z}_{[N-1]}. \tag{12b}$$

Consider now the constraint (4e). In order to guarantee satisfaction of this constraint by the constructed candidate state trajectory, the triggering set $\mathcal{E}_{j,k}$ should respect the constraint $x^*_{k+j+i|k} \oplus \tilde{L}_i\mathcal{E}_{j,k} \subset \mathcal{X}_i$, for all $i \in \mathsf{Z}_{[N-1]}$. With regards to the constraint (4d), one can use similar arguments to arrive at $u^*_{k+j+i|k} \oplus \tilde{K}_i\tilde{L}_i\mathcal{E}_{j,k} \subset \mathcal{U}_i$, for all $i \in \mathsf{Z}_{[N-1]}$. (Let us also remark that the satisfaction of the constraint (4f) follows from the nilpotency of the tightening gains $\mathbf{K}$. See [12] for more details.)

In the last step, our goal is to find a quantitative way to capture the cost associated with the constructed candidate trajectories. To this end, we introduce two new sets of parameters representing the distance of the optimal input and state trajectories from the target sets (6c)-(6d). Observe that

$$s^*_{x,k+j+i|k} := \underset{s_x \in \mathcal{T}_{x,j+i}}{\text{argmin}}\ \|x^*_{k+j+i|k} - s_x\|^2_Q,\ \text{ if } j+i \leq N, \tag{13a}$$

$$s^*_{u,k+j+i|k} := \underset{s_u \in \mathcal{T}_{u,j+i}}{\text{argmin}}\ \|u^*_{k+j+i|k} - s_u\|^2_R,\ \text{ if } j+i \leq N-1, \tag{13b}$$

are available from solving $\mathcal{P}(x_k)$. Recall now the convention we introduced in (11). Based on this convention and in light of the relations (7), we define for all $i \in \mathsf{Z}_{[N-1]}$,

$$s^*_{x,k+j+i|k} := x^*_{k+j+i|k}, \text{ if } j+i \in \mathsf{Z}_{[N+j]}/\mathsf{Z}_{[N]}, \quad (14a)$$

$$s^*_{u,k+j+i|k} := u^*_{k+j+i|k}, \text{ if } j+i \in \mathsf{Z}_{[N+j-1]}/\mathsf{Z}_{[N-1]}. \quad (14b)$$

The parameters defined in (13) and (14) enable us to identify the conditions on the triggering set $\mathcal{E}_{j,k}$ by which the cost function (5) corresponding to the candidate trajectories is "well-behaved". In particular, the cost function is guaranteed to decrease (respectively, not increase) when the states and inputs are outside (respectively, inside) their corresponding target sets. See [12] for more details.

We are now ready to introduce all the set-type inequalities that $\mathcal{E}_{j,k}$ should satisfy. These conditions are

$$\underline{e}^p_{j,k} \geq 0, \ \bar{e}^p_{j,k} \geq 0, \qquad \forall p \in \mathsf{N}_{[n_x]}, \quad (15a)$$

$$x^*_{k+j+i|k} \in \mathcal{X}_i \sim \tilde{L}_i \mathcal{E}_{j,k}, \qquad \forall i \in \mathsf{Z}_{[N-1]}, \quad (15b)$$

$$u^*_{k+j+i|k} \in \mathcal{U}_i \sim \tilde{K}_i \tilde{L}_i \mathcal{E}_{j,k}, \qquad \forall i \in \mathsf{Z}_{[N-1]}, \quad (15c)$$

$$s^*_{x,k+j+i|k} \in \mathcal{T}_{x,i} \sim \tilde{L}_i \mathcal{E}_{j,k}, \qquad \forall i \in \mathsf{Z}_{[N-1]}, \quad (15d)$$

$$s^*_{u,k+j+i|k} \in \mathcal{T}_{u,i} \sim \tilde{K}_i \tilde{L}_i \mathcal{E}_{j,k}, \qquad \forall i \in \mathsf{Z}_{[N-1]}. \quad (15e)$$

Notice that the constraint (15a) identifies all the states $x_{k+j} \in x^*_{k+j|k} \oplus \mathcal{E}_{j,k}$.

B. Event-Triggered Robust MPC

We have so far provided the conditions that each $\mathcal{E}_{j,k}$ should satisfy. It remains to select a suitable objective function to construct this set. A natural choice is the volume of $\mathcal{E}_{j,k}$, $\text{vol}(\mathcal{E}_{j,k}) := \Pi_{p \in \mathsf{N}_{[n_x]}} (\bar{e}^p_{j,k} + \underline{e}^p_{j,k})$. To construct $\mathcal{E}_{j,k}$, we hence consider $\text{vol}(\mathcal{E}_{j,k})$ as the objective function along with the constraints (15a)-(15e). Moreover, we borrow some results from [5] to propose an LP relaxation of the problem. In order to avoid heavily involved notations, we also state the result in terms of some general parameters.

Theorem 3.1 (LP relaxed construction): Consider a vector $\xi \in \mathbb{R}^p$, a matrix $M \in \mathbb{R}^{p \times k}$, and a polytope $\mathcal{S} = \{s \in \mathbb{R}^p : \ A_{\mathcal{S}} s \leq b_{\mathcal{S}}\}$ containing the origin where $A_{\mathcal{S}} \in \mathbb{R}^{m \times p}$ and $b_{\mathcal{S}} \in \mathbb{R}^m$. The maximum volume r-constrained hyper-rectangle $\mathcal{B}(l,u) \subset \mathbb{R}^k$ that contains the origin and satisfies $\xi \in \mathcal{S} \sim M\mathcal{B}(l,u)$ is $\mathcal{B}(z^*, z^* + \lambda^* r)$ where

$$\begin{aligned}(z^*, \lambda^*) := \ & \underset{z,\lambda}{\text{argmax}} \ \lambda \\ & \text{s.t. } A_{\mathcal{S}} M z + (A_{\mathcal{S}} M)^+ r \lambda \leq b_{\mathcal{S}} - A_{\mathcal{S}} \xi \\ & \quad z + \lambda r \geq 0, \ z \leq 0,\end{aligned} \quad (16)$$

where for all $j \in \mathsf{N}_{[k]}$, the j-th entry of r is defined as

$$\begin{aligned} r_j := \ & \max_{z,\omega} \ \omega \\ & \textit{s.t.} \ A_{\mathcal{S}} M z \leq b_{\mathcal{S}} - A_{\mathcal{S}} \xi \\ & \quad A_{\mathcal{S}} M (z + \omega e_j) \leq b_{\mathcal{S}} - A_{\mathcal{S}} \xi \\ & \quad z + \omega e_j \geq 0, \ z \leq 0, \end{aligned} \quad (17)$$

where $e_j \in \mathbb{R}^k$ is the unit vector in the j-th direction.

Proof: See [12]. ■

Suppose now each set $\mathcal{E}_{j,k}$ in the triggering mechanism (9) is constructed based on Theorem 3.1. We now state the results about the recursive feasibility and the robust stability of the RMPC method (4) using the triggering mechanism (9).

Theorem 3.2 (Recursive feasibility): Consider the constrained dynamics (1)-(2). Suppose the initial state $x_0 \in \mathbb{R}^{n_x}$ is such that $\mathcal{P}(x_0)$ is feasible. Then, the state and input trajectories of the dynamics (1) with the triggering mechanism (9) satisfy the constraints (2), for all $k \in \mathbb{Z}_{\geq 0}$, i.e., robust recursive feasibility.

Proof: See [12]. ■

Theorem 3.3 (Robust convergence): Consider the constrained dynamics (1)-(2). Suppose the initial state $x_0 \in \mathbb{R}^{n_x}$ is such that $\mathcal{P}(x_0)$ is feasible. Then, the state and input trajectories of dynamics (1) with the triggering mechanism (9) are such that $x_k \to \mathbb{T}_x$ and $u_k \to \mathbb{T}_u$, as $k \to \infty$, i.e., robust convergence.

Proof: See [12]. ■

Let us close this section with highlighting two properties of the proposed event-triggering design. First, observe that the proposed approach is online. As a result, at each triggering instant it is required to calculate the triggering sets. However, the extra design problem is a linear program that has a lower complexity with respect to the complexity of the RMPC problem. (An RMPC problem with a quadratic cost is a quadratic program.)

Second, we note the difference between the proposed approach in this paper and the Lebesgue-type sampling approaches proposed in the literature. To the best of our knowledge, most of those approaches define a ball around the optimal state trajectory based on a certain norm. Then, they study the impact of the radius of this ball on feasibility and convergence of the closed-loop dynamics. However, the proposed approach in this paper does not necessarily construct triggering sets (hyper-rectangles) that are symmetric with respect to the computed optimal state trajectory at the last triggering instant.

IV. NUMERICAL EXAMPLE

We now provide a numerical example to show the effectiveness of results presented in this paper. Consider the perturbed LTI system

$$x_{k+1} = \begin{pmatrix} 1 & 0.1 \\ 0 & 1 \end{pmatrix} x_k + \begin{pmatrix} 0.005 \\ 0.1 \end{pmatrix} u_k + w_k,$$

where the states and input constraint sets are $\mathbb{X} = \{x \in \mathbb{R}^2 : \ |x_1| \leq 5, \ |x_2| \leq 3\}$ and $\mathbb{U} = \{u \in \mathbb{R} : \ |u| \leq 5\}$, respectively. The state and input target sets are $\mathbb{T}_x = \{x \in \mathbb{R}^2 : \ \|x\|_\infty \leq 1\}$ and $\mathbb{T}_u = \{u \in \mathbb{R} : \ |u| \leq 3\}$, respectively. The weight matrices in the cost function (5) are

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 0.001 \end{pmatrix} \text{ and } R = 10.$$

The terminal set is $\mathcal{X}_f = \{x \in \mathbb{R}^2 : \ \|x\|_\infty \leq 0.1\}$.

We first show the result of using Theorem 3.1 to construct the triggering sets $\mathcal{E}_{j,k}$. As we have pointed out at the end of Section III, the construction method proposed in this paper results in triggering sets that are not symmetric with respect to the optimal state trajectory computed at the last triggering instant.

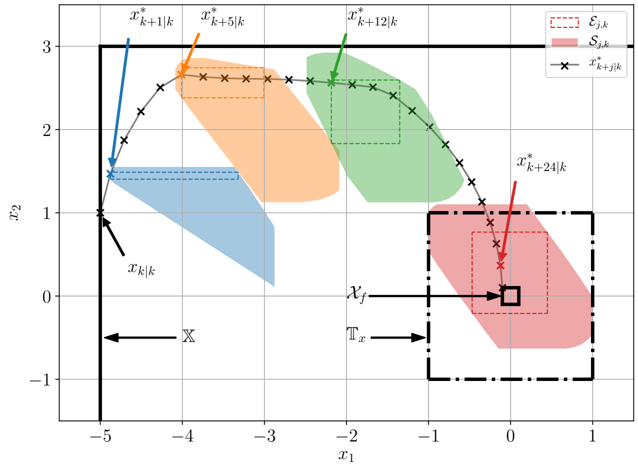


Fig. 1: Constructed hyper-rectangles $\mathcal{E}_{j,k}$ for $j = 1, 5, 12, 24$: the horizon length $N = 25$, the nilpotency length $M = 24$, and the disturbance set $\mathcal{W} = \{w \in \mathbb{R}^2 : \; w_1 = 0, |w_2| \leq 0.15\}$.

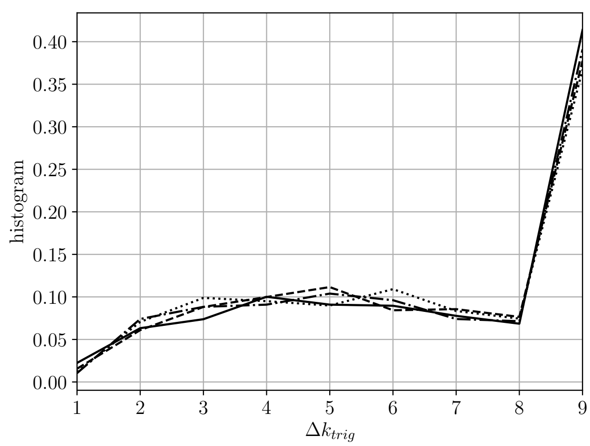


Fig. 2: Impact of initial state on the distribution of inter-execution times: the horizon length $N = 10$, the nilpotency length $M = 9$, and the disturbance set $\mathcal{W} = \{w \in \mathbb{R}^2 : \; w_1 = 0, |w_2| \leq 0.1\}$.

In Fig.2, the relative frequency, or the probability mass functions (pmf's), of the inter-execution times Δk_{trig} for four equidistant initial conditions x_0 on the ball $\|x_0\| = 0.5$ are depicted. Notice that the pmf's for all initial states almost converge to a similar distribution. Indeed, our simulations show that the sequence of inter-execution times converge to a *stationary* state in long-term.

We next study the impact of the horizon length N on the pmf(Δk_{trig}) and the expectation $\mathbb{E}(\Delta k_{\text{trig}})$ in Fig. 3. It is evident that as N increases, $\mathbb{E}(\Delta k_{\text{trig}})$ will also increase. Recall that as the horizon length increases the computation costs of the RMPC method and the triggering mechanism design increase. Hence, one should consider a trade-off between the computational cost of having a larger horizon length and the possibility of reducing the number of triggering instants.

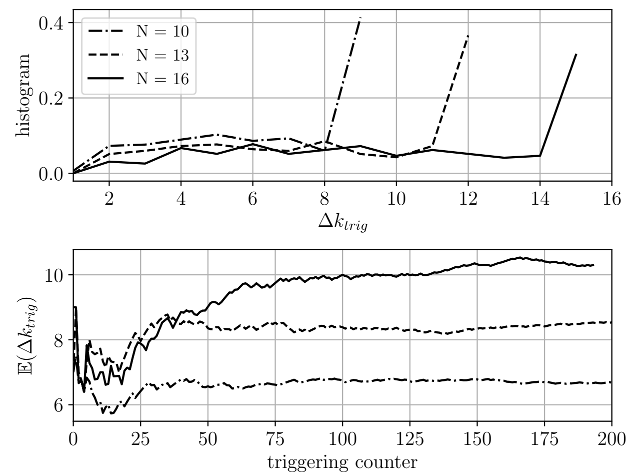


Fig. 3: Impact of horizon length N on the distribution of inter-execution times (top) and on the expectation of inter-execution times (bottom): $N = 10, 13, 16$, the nilpotency length $M = N - 1$ for each N, and the disturbance set $\mathcal{W} = \{w \in \mathbb{R}^2 : \; w_1 = 0, |w_2| \leq 0.15\}$.

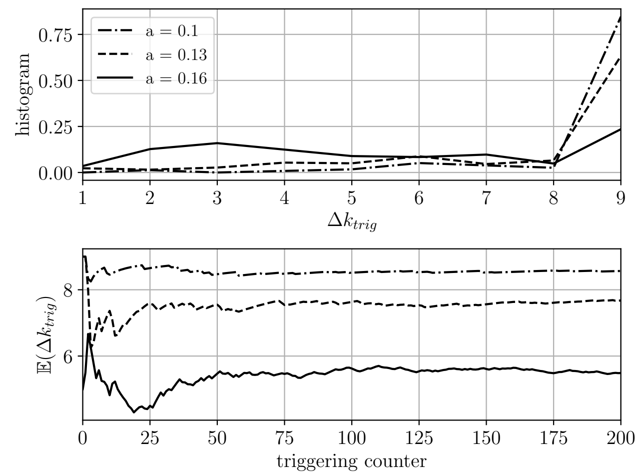
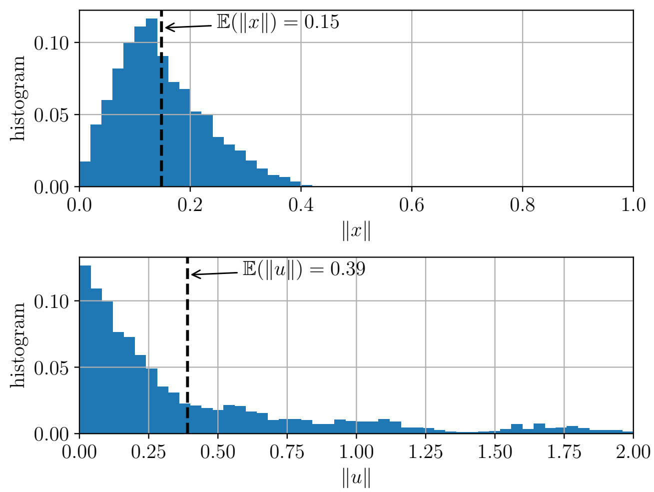


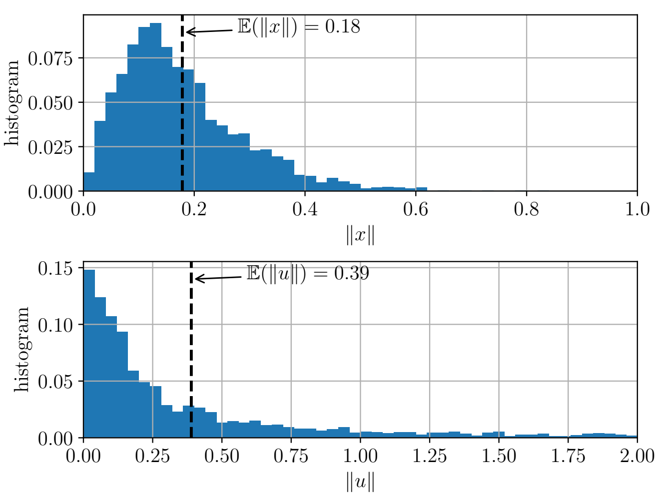
Fig. 4: Impact of disturbance on the distribution of inter-execution times (top) and on the expectation of inter-execution times (bottom): the horizon length $N = 10$, the nilpotency length $M = 9$, and the disturbance set $\mathcal{W} = \{w \in \mathbb{R}^2 : \; w_1 = 0, |w_2| \leq a\}$ where $a = 0.1, 0.13, 0.16$.

In Fig. 4, the impacts of disturbance on the pmf(Δk_{trig}) and the expectation $\mathbb{E}(\Delta k_{\text{trig}})$ are reported. It is interesting to observe that as the size of $\mathcal{W}$ increases, the distribution of Δk_{trig} approaches a uniform one (see Fig. 4, top). This in turn results in a decrease of the expectation of inter-execution times as depicted in Fig. 4, bottom.

We finally compare the behavior of the event-triggered RMPC method with the standard RMPC method (4) where we let the the standard method to run in an open loop fashion



(a) Event-triggered implementation.



(b) Standard method with the open-loop implementation

Fig. 5: Comparison of event-triggered and time-triggered cases: the horizon length $N = 10$, the nilpotency length $M = 9$, and the disturbance set $\mathcal{W} = \{w \in \mathbb{R}^2 : \ w_1 = 0, |w_2| \leq 0.15\}$.

for $\lfloor \mathbb{E}(\Delta k_{\text{trig}}) \rfloor$ instants. (Notice that in general the open-loop implementation is not guaranteed to remain stable and feasible.) Fig. 5 represents the pmf and expectation of $\|x\|$ and $\|u\|$. One can observe that the event-triggered case has better statistical properties.

V. Conclusions

In this paper, a linear programming approach to construct an online triggering mechanism for a robust MPC method is proposed. The proposed approach can be seen as a Lebesgue-type sampling. Unlike most of the approaches in literature, the Lebesgue thresholds are time-varying over the horizon, and depend on the dynamics at the triggering instant and the constraints on the dynamics. We have conducted several numerical experiments to show the effectiveness of our proposed approach. As the closed-loop system reaches its steady-state, the triggering mechanism significantly reduces the number of times at which the RMPC problem should be solved. Since the design of the triggering mechanism is online, the computational cost of the proposed event-triggered method at the triggering instants is higher compared to the standard robust MPC. Therefore, a logical future research direction is to make this design process offline.

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