Robust Control Design for Linear Systems
via Multiplicative Noise[*]

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Abstract:
Robust stability and stochastic stability have separately seen intense study in control theory for many decades. In this work we establish relations between these properties for discrete-time systems and employ them for robust control design. Specifically, we examine a multiplicative noise framework which models the inherent uncertainty and variation in the system dynamics which arise in model-based learning control methods such as adaptive control and reinforcement learning. We provide results which guarantee robustness margins in terms of perturbations on the nominal dynamics as well as algorithms which generate maximally robust controllers.

Keywords: Robust controller synthesis, robust control (linear case), uncertainty descriptions, stochastic systems.

1. INTRODUCTION

Model-based learning control, which encompasses classical system identification (e.g. Ljung (2001)) and adaptive control (e.g. Åström and Wittenmark (2013)) as well as branches of modern reinforcement learning (e.g. Nagabandi et al. (2018); Tu and Recht (2019)), universally uses a stochastic data model, where a model is estimated from data corrupted by random noise. A salient perennial issue in these methods is ensuring stability despite the presence of concomitant model errors; this is the problem of *robustness*.

Traditional methods for designing robust controllers include $\mathcal{H}_\infty$ control design, which treats modeling error as a worst-case or adversarial disturbance (Zhou and Doyle (1998); Başar and Olsder (1998)), robust optimization over parametric state-space uncertainty sets, which typically involve searching for shared Lyapunov functions via convex semidefinite programming (Khargonekar et al. (1990); Nemirovskii (1993); Boyd et al. (1994); Corless (1994); De Oliveira et al. (1999)), and certainty-equivalent control, which utilizes only a nominal model and ignores modeling error entirely. However, since the robust design methods work with uncertainty sets, it is generally not straightforward to relate the uncertainty set descriptions to actual uncertainties arising from a stochastic data model.

Alternatively, in this paper we explore the connection between a special type of *stochastic stability* and *robust stability* and exploit this connection for robust control design. In particular, we use a *multiplicative noise* model where the noise is viewed as a representation of uncertainty

in the nominal system model. This framework is naturally disposed toward trading off performance and robustness according to uncertainty directions and magnitudes which can be estimated from trajectory data during model-based learning control. The study of multiplicative noise models, also known as stochastic parameter models, has a long history in control theory (Kushner (1967); Wonham (1967); Kozin (1969)). In contrast with the well-known additive noise setting, multiplicative noise captures linear dependence of the noise on the state and control input, which occurs intrinsically in a diverse array of modern control systems such as robotics (Du Toit and Burdick (2011)), networked systems with noisy communication channels (Hespanha et al. (2007)), modern power networks with high penetration of intermittent renewables (Carrasco et al. (2006)), turbulent fluid flow (Lumley (2007)). Linear systems with multiplicative noise are particularly attractive as a stochastic modeling framework because they remain simple enough to admit closed-form expressions for stability and optimal control via generalized Lyapunov and Riccati equations. A multiplicative noise model also holds a distinct advantage of being sensitive to *structured* uncertainties in *specific directions directly related to data*, as opposed to generic sets governed by norm balls as in Dean et al. (2019).

In this paper we consider a fundamental question:

What is the set of perturbations to the system matrix where the perturbed system can be guaranteed stable, given knowledge only of the nominal system dynamics and stochastic stability of a system with multiplicative noise?

This question was considered by Bernstein (1987) for the continuous-time setting. Surprisingly, it was noted that the addition of multiplicative noise could actually *stabilize*

a deterministically unstable system when interpreted in the sense of Stratonovich (rather than Itô) Arnold et al. (1983). Despite this difficulty, combining mean-square stability of a multiplicative noise system with a *right-shift* of the system dynamics, i.e., increasing the real parts of the eigenvalues of A as $A \leftarrow A + cI$ was shown sufficient to ensure robust deterministic stability. Similarly, we develop conditions for discrete-time systems which combine mean-square stability of a multiplicative noise system with a *scaling* of the system dynamics, i.e., increasing the absolute value of eigenvalues of A as $A \leftarrow cA$.

In this paper we make the following contributions:

- We develop a result utilizing shared Lyapunov functions that establishes robust stability of a set of deterministic systems given stochastic (mean-square) stability of another system with multiplicative noise (Theorem 7).
- We develop a complementary result utilizing an auxiliary system with scaled dynamics matrices that establishes robust stability of a set of deterministic systems given stochastic (mean-square) stability of another system with multiplicative noise (Theorem 14).
- We show that both theorems yield robustness sets whose size increases monotonically with the multiplicative noise variances and collapse to zero in the case of zero noise.
- We develop a corresponding pair of algorithms which efficiently compute controllers that simultaneously maximize robustness and minimize a quadratic cost.

We elaborate on the robust stability problem in Section 2, develop theorems in Sections 3 and 4, develop corresponding algorithms in Section 5, give numerical examples in Section 6, and conclude in Section 7.

2. PROBLEM FORMULATION

Consider the dynamics of a discrete-time linear time-invariant (LTI) system

$$x_{t+1} = \bar{A}x_t + \bar{B}u_t \qquad (1)$$

where the entries of $\bar{A}$ and $\bar{B}$ are unknown constants and are approximated (perhaps from noisy trajectory data) by the known nominal matrices A and B leading to the nominal model

$$x_{t+1} = Ax_t + Bu_t \qquad (2)$$

where $x_t \in \mathbb{R}^n$ is the system state, $u_t \in \mathbb{R}^m$ is the control input, $A \in \mathbb{R}^{n \times n}$ is the dynamics matrix and $B \in \mathbb{R}^{n \times m}$ is the input matrix. In order to stabilize the system in (2), we use linear state feedback $u_t = Kx_t$ with gain matrix $K \in \mathbb{R}^{m \times n}$; classical results Kalman et al. (1960, 1963) show that if the pair (A, B) is controllable, then the closed-loop eigenvalues of $A + BK$ can be placed arbitrarily by choosing suitable gains. A robust stabilization problem is to find a linear state-feedback control $u_t = Kx_t$ such that the closed-loop nominal system remains stable under fixed perturbations of A and B i.e. that

$$x_{t+1} = \big[(A + \Delta A) + (B + \Delta B)K\big]x_t \qquad (3)$$

is stable for some set of perturbations $\Delta A \in \mathcal{A}$ and $\Delta B \in \mathcal{B}$, ideally containing the true matrices $\bar{A}$ and $\bar{B}$.

As a parallel development, consider an LTI system with multiplicative noise with dynamics

$$x_{t+1} = \Big(A + \sum_{i=1}^{p}\gamma_{ti}A_i\Big)x_t + \Big(B + \sum_{j=1}^{q}\delta_{tj}B_j\Big)u_t \qquad (4)$$

Multiplicative noise terms are modeled by the i.i.d. across time (white), zero-mean, mutually independent scalar random variables γ_{ti} and δ_{tj}, which have variances α_i and β_j, respectively. The matrices $A_i \in \mathbb{R}^{n \times n}$ and $B_i \in \mathbb{R}^{n \times m}$ specify how each scalar noise term affects the dynamics and input matrices.

Stability of such a system depends on the behavior of the second moments (covariance) of the state over time as formalized by the notion of *mean-square stability*, a form of robust stability which is stricter than stabilizability of the nominal system (A, B) and limits the size of the multiplicative noise variances (Kozin (1969); Willems and Willems (1976)):

Definition 1. (Mean-square stability). The system in (4) is mean-square stable if and only if

$$\lim_{t \to \infty} \mathbb{E}\big[x_t x_t^\mathsf{T}\big] = 0, \qquad \forall x_0 \in \mathbb{R}^n$$

In order to stabilize the system in (2), we again use linear state feedback $u_t = Kx_t$; mean-square stability of the closed-loop system with this control is equivalently characterized by the solution of a *generalized Lyapunov equation* (GLE) (Kleinman (1969); Boyd et al. (1994)):

Lemma 2. The system in (4) is mean-square stable in closed-loop with state feedback $u_t = Kx_t$ if and only if for any $Q \succ 0$ there exists $P \succ 0$ satisfying

$$P = Q + (A + BK)^\mathsf{T}P(A + BK)$$
$$+ \sum_{i=1}^{p}\alpha_i A_i^\mathsf{T}PA_i + \sum_{j=1}^{q}\beta_j K^\mathsf{T}B_j^\mathsf{T}PB_j K. \qquad (5)$$

This immediately implies the following:

Corollary 3. In the discrete-time setting, mean-square stability of (4) with control $u_t = Kx_t$ implies deterministic stability of (2) with the same control $u_t = Kx_t$.

Proof. From (5), strict mean-square stability implies existence of $P \succ 0$ such that

$$P = Q + (A + BK)^\mathsf{T}P(A + BK)$$
$$+ \sum_{i=1}^{p}\alpha_i A_i^\mathsf{T}PA_i + \sum_{j=1}^{q}\beta_j K^\mathsf{T}B_j^\mathsf{T}PB_j K$$
$$\succeq Q + (A + BK)^\mathsf{T}P(A + BK)$$

which ensures stability of $A + BK$.

One mean-square stabilizing control arises from solving the infinite-horizon LQR problem with multiplicative noise

$$\min_{\pi} \quad \mathbb{E}_{\{\gamma_{ti}\},\{\delta_{tj}\}} \sum_{t=0}^{\infty}(x_t^\mathsf{T}Qx_t + u_t^\mathsf{T}Ru_t),$$

$$\text{s.t.} \quad x_{t+1} = \Big(A + \sum_{i=1}^{p}\gamma_{ti}A_i\Big)x_t + \Big(B + \sum_{j=1}^{q}\delta_{tj}B_j\Big)u_t,$$

where $Q \succeq 0$, $R \succ 0$ and inputs are applied according to the state-feedback policy $u_t = \pi(x_t)$. We assume that the problem data A, B, α_i, A_i, β_j, and B_j permit the existence of a finite solution, in which case the system is called *mean-square stabilizable*. Dynamic programming can be used to show that the optimal policy is linear state feedback

$u_t = K^* x_t$, where $K^* \in \mathbb{R}^{m \times n}$ denotes the optimal gain matrix, and the resulting optimal cost $V(x_0)$ for a fixed initial state x_0 is quadratic, i.e., $V(x_0) = x_0^\intercal P x_0$, where $P \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix. There are several ways to compute the optimal feedback gains and corresponding optimal cost. The optimal cost is given by the solution of the *generalized algebraic Riccati equation* (GARE)

$$P = Q + A^\intercal P A + \sum_{i=1}^{p} \alpha_i A_i^\intercal P A_i$$
$$- A^\intercal P B (R + B^\intercal P B + \sum_{j=1}^{q} \beta_j B_j^\intercal P B_j)^{-1} B^\intercal P A$$

which can be derived similarly to the GARE given by McLane (1972) for continuous-time systems. The solution $P = \mathrm{gare}(A, B, Q, R, \alpha_i, \beta_j, A_i, B_j)$ can be obtained via the recursion

$$P_{t+1} = Q + A^\intercal P_t A + \sum_{i=1}^{p} \alpha_i A_i^\intercal P_t A_i$$
$$- A^\intercal P_t B (R + B^\intercal P_t B + \sum_{j=1}^{q} \beta_j B_j^\intercal P_t B_j)^{-1} B^\intercal P_t A,$$

with $P_0 = Q$ or via semidefinite programming formulations (Boyd et al. (1994); El Ghaoui (1995); Li et al. (2005)). The corresponding optimal gain matrix is then

$$K^* = -\left(R + B^\intercal P B + \sum_{j=1}^{q} \beta_j B_j^\intercal P B_j\right)^{-1} B^\intercal P A.$$

2.1 Generalized eigenvalues and semidefiniteness

The following lemmas regarding generalized eigenvalue problems and semidefiniteness will be needed later:

Lemma 4. Let M and N be positive semidefinite matrices of the same size. If $\lambda_{\max}$ is the maximum generalized eigenvalue which solves $Mv = \lambda N v$, then $\lambda_{\max} N \succeq M$.

Corollary 5. If N is singular, then $\lambda_{\max}$ becomes infinite.

We omit the proofs since these results are widely known; they follow readily from the method of Lagrange multipliers and Rayleigh quotients.

Every symmetric matrix S can be split into positive and negative semidefinite parts via eigendecomposition as

$$S = S^+ + S^-$$

where

$$S^+ = \sum_i \lambda_i v_i v_i^\intercal \succeq S \succeq 0, \quad S^- = \sum_j \lambda_j v_j v_j^\intercal \preceq S \preceq 0,$$

where λ_i and λ_j are positive and negative eigenvalues respectively with associated eigenvectors v_i and v_j.

3. ROBUSTNESS VIA SHARED LYAPUNOV FUNCTIONS

We begin by ignoring the contribution of feedback control; we will introduce the control again in Sec. 5. We also restrict our search over ΔA to the set

$$\Delta A \in \mathcal{A} = \left\{ \sum_{i=1}^{p} \mu_i A_i \ \Big| \ \mu_i \in \mathbb{R}, 0 \le \mu_i < y\theta_i, \sum_{i=1}^{p} \theta_i = 1 \right\}$$

The θ_i are scalars that represent the relative amount of uncertainty in each direction, while y is a scalar governing the maximum magnitude of the perturbations. The θ_i and y can be estimated from statistics of sampled trajectory data, e.g., using bootstrap resampling methods. This approach is intuitive; mean-square stability under stochastic instantaneous perturbations in specific directions A_i ought to ensure deterministic stability under constant shifts of the dynamics in those same directions. Note the number of linearly independent uncertainty directions p is limited by the number of entries of A i.e. $p \le n^2$. Consider the problem of finding the largest deviation scalar y^* which can be tolerated while still guaranteeing stability of the perturbed deterministic system

$$x_{t+1} = (A + \Delta A)x_t, \ \Delta A \in \mathcal{A}$$

based on mean-square stability of the stochastic system

$$x_{t+1} = (A + \gamma_{ti} A_i)x_t$$

with $\mathbb{E}[\gamma_{ti}] = 0$, $\mathbb{E}[\gamma_{ti}^2] = \alpha_i > 0$.

3.1 Scalar case

First, we treat the scalar case where $n = p = 1$ so $A_1 = 1$ and $\theta_1 = 1$ without loss of generality.

Lemma 6. Suppose

$$x_{t+1} = (A + \gamma_t)x_t$$

is mean-square stable where A, x_t, γ_t are scalars with $\mathbb{E}[\gamma_t^2] = \alpha > 0$. Then, the perturbed deterministic system

$$x_{t+1} = (A + y)x_t \tag{6}$$

is stable for any fixed perturbation $|y| \le \sqrt{A^2 + \alpha} - |A|$.

Proof. The GLE in (5) reduces to

$$P = Q + A^2 P + \alpha P$$

where P, Q are scalars with solution

$$P = Q \left[1 - (A^2 + \alpha) \right]^{-1}$$

which is positive only when $\sqrt{A^2 + \alpha} < 1$. By assumption the system is mean-square stable, so Lemma 2 implies that the solution $P > 0$ and thus indeed $\sqrt{A^2 + \alpha} < 1$. By the restriction on y and the triangle inequality

$$\rho(A + y) = |A + y| \le |A| + |y| \le \sqrt{A^2 + \alpha} < 1,$$

proving stability of (6).

This simple example demonstrates that the robustness margin increases monotonically with the multiplicative noise variance and when $\alpha = 0$, i.e. $|A| \to 1$, the bound collapses and no robustness is guaranteed.

3.2 Multivariate case

The optimal bound y^* is found by solving the program

$$\underset{y,P}{\text{maximize}} \quad y$$

$$\text{subject to} \quad P \succeq I + A^\intercal P A + \sum_{i=1}^{p} \alpha_i A_i^\intercal P A_i$$
$$P \succeq (A + \Delta A)^\intercal P (A + \Delta A) \tag{7}$$
$$\Delta A = y \sum_{i=1}^{p} k_i \theta_i A_i \ \forall \ k_i \in \{-1, +1\}$$

i.e. maximizing y while ensuring that there exists a P which generates a Lyapunov function which guarantees

both mean-square stability of the stochastic system and deterministic stability of the perturbed deterministic system. Here we have arbitrarily chosen $Q = I$ e.g. as in (5) without loss of generality since the constraints pertain only to stability, which is invariant to the choice of Q. Since the program is quasiconvex in y, it can be solved by bisection over y and solving a feasibility SDP for each fixed y, with the solution being the largest y which admits a feasible solution to the SDP.

The set of constraints in the second line of (7) form corners of a convex box polytope in the space of $n \times n$ matrices, which is necessary and sufficient to guarantee stability of (3) (Boyd et al. (1994); Corless (1994)). Thus, from the perspective of verifying stability of $A + \Delta A$ this procedure no better than simply solving the same program (7) with the first constraint deleted, which has a larger feasible set and thus will achieve at least as good a bound as (7). However, the solution of (7) defines a hard upper limit on the following bounds we develop in this section which are based on a shared Lyapunov function, since (7) gives the optimal bound. The bounds we develop in this section trade optimality (conservativeness) for the assurance that P guarantees stability of the perturbed deterministic system without explicitly using the Lyapunov inequality $P \succeq (A + \Delta A)^\intercal P(A + \Delta A)$.

Giving up optimization over P and instead choosing Q arbitrarily (later in Sec. 5, Q will be chosen as the cost matrix of an LQR control design) and calculating the associated P, we obtain the following result:

Theorem 7. Suppose

$$x_{t+1} = \left(A + \sum_{i=1}^{p} \gamma_{ti} A_i\right) x_t \tag{8}$$

is mean-square stable with $\mathbb{E}[\gamma_{ti}] = 0$, $\mathbb{E}[\gamma_{ti}^2] = \alpha_i > 0$. Fix a $Q \succeq I$ and the solution $P \succ 0$ to

$$P = pQ + A^\intercal PA + \sum_{i=1}^{p} \alpha_i A_i^\intercal PA_i \tag{9}$$

Let $\eta_i > 0$ be scalars which satisfy

$$pQ + \sum_{i=1}^{p} \alpha_i A_i^\intercal PA_i \succeq \sum_{i=1}^{p} \eta_i \left(A_i^\intercal PA + A^\intercal PA_i\right)^+ \tag{10}$$
$$+ \sum_{i=1}^{p} \sum_{j=1}^{p} \eta_i \eta_j \left(A_i^\intercal PA_j + A_j^\intercal PA_i\right)^+$$

Then the deterministic system

$$x_{t+1} = \left(A + \sum_{i=1}^{p} \mu_i A_i\right) x_t \tag{11}$$

is deterministically stable for any

$$0 \le \mu_i < \eta_i \tag{12}$$

Proof. It is evident that valid $\eta_i > 0$ exist since $pQ + \sum_{i=1}^{p} \alpha_i A_i^\intercal PA_i$ is strictly positive definite. Rearranging (9) to $pQ + \sum_{i=1}^{p} \alpha_i A_i^\intercal PA_i = P - A^\intercal PA$ and substituting gives

$$P \succeq A^\intercal PA + \sum_{i=1}^{p} \eta_i \left(A_i^\intercal PA + A^\intercal PA_i\right)^+$$
$$+ \sum_{i=1}^{p} \sum_{j=1}^{p} \eta_i \eta_j \left(A_i^\intercal PA_j + A_j^\intercal PA_i\right)^+$$
$$\succeq A^\intercal PA + \sum_{i=1}^{p} \mu_i \left(A_i^\intercal PA + A^\intercal PA_i\right)^+$$
$$+ \sum_{i=1}^{p} \sum_{j=1}^{p} \mu_i \mu_j \left(A_i^\intercal PA_j + A_j^\intercal PA_i\right)^+$$
$$\succeq A^\intercal PA + \sum_{i=1}^{p} \mu_i \left(A_i^\intercal PA + A^\intercal PA_i\right)$$
$$+ \sum_{i=1}^{p} \sum_{j=1}^{p} \mu_i \mu_j \left(A_i^\intercal PA_j + A_j^\intercal PA_i\right)$$
$$= \left(A + \sum_{i=1}^{p} \mu_i A_i\right)^\intercal P \left(A + \sum_{i=1}^{p} \mu_i A_i\right)$$

which proves stability of (11).

Remark 8. The bound in (12) of Thm. 7 is unidirectional, but can be made bidirectional by replacing the constraint in (10) with

$$pQ + \sum_{i=1}^{p} \alpha_i A_i^\intercal PA_i \succeq \sum_{i=1}^{p} \eta_i Y_i + \sum_{i=1}^{p} \sum_{j=1}^{p} \eta_i \eta_j Z_{ij}$$

where

$$Y_i \succeq \left(A_i^\intercal PA + A^\intercal PA_i\right)^+, \tag{13}$$
$$Y_i \succeq -\left(A_i^\intercal PA + A^\intercal PA_i\right)^-, \tag{14}$$
$$Z_{ij} \succeq \left(A_i^\intercal PA_j + A_j^\intercal PA_i\right)^+, \tag{15}$$
$$Z_{ij} \succeq -\left(A_i^\intercal PA_j + A_j^\intercal PA_i\right)^-, \tag{16}$$

yielding the bidirectional bound $|\mu_i| < \eta_i$.

Remark 9. Let $\theta_i \ge 0$ be scalars such that $\sum_{i=1}^{p} \theta_i = 1$; these denote relative uncertainty in directions A_i. The largest robust stability bounds with respect to this choice of θ_i are obtained by setting $\eta_i = y\theta_i$ and maximizing the scalar y, which can be accomplished via bisection. As discussed earlier, optimizing a bidirectional bound over P, Q, and y is equivalent to solving the full program in (7).

For $p = 1$, the Theorem 7 reduces as follows:

Corollary 10. Suppose

$$x_{t+1} = (A + \gamma_{t1} A_1) x_t$$

is mean-square stable with $\mathbb{E}[\gamma_{t1}] = 0$, $\mathbb{E}[\gamma_{t1}^2] = \alpha_1 > 0$. Fix a $Q \succeq I$ and the solution $P \succ 0$ to

$$P = Q + A^\intercal PA + \alpha_1 A_1^\intercal PA_1$$

Let $\zeta_1 > 0$ be a scalar which satisfies

$$\frac{1}{\sqrt{\zeta_1^2 + \alpha_1} - \zeta_1} Q + 2\zeta_1 A_1^\intercal PA_1 \succeq \left(A^\intercal PA_1 + A_1^\intercal PA\right)^+. \tag{17}$$

Then the deterministic system

$$x_{t+1} = (A + \mu_1 A_1) x_t$$

is deterministically stable for any

$$0 \le \mu_1 < \eta_1$$

where $\eta_1 > 0$ is a scalar uniquely determined by ζ_1 as

$$\eta_1 = \sqrt{\zeta_1^2 + \alpha_1} - \zeta_1 \ (\leq \sqrt{\alpha_1})$$

Also, η_1 satisfies

$$Q + \alpha_1 A_1^\intercal P A_1 \succeq \eta_1 \left(A_1^\intercal PA + A^\intercal PA_1\right)^+ + 2\eta_1^2 A_1^\intercal P A_1 \tag{18}$$

in accordance with Thm. 7.

Proof. Multiplying both sides of (17) by η_1 and using $\eta_1 = \sqrt{\zeta_1^2 + \alpha_1} - \zeta_1$ gives

$$\begin{aligned}
Q + 2\eta_1\zeta_1 A_1^\intercal P A_1 &= \frac{\eta_1}{\sqrt{\zeta_1^2 + \alpha_1} - \zeta_1}Q + 2\eta_1\zeta_1 A_1^\intercal P A_1 \\
&\succeq \eta_1\left(A^\intercal P A_1 + A_1^\intercal PA\right)^+
\end{aligned} \tag{19}$$

Rearranging $\eta_1 = \sqrt{\zeta_1^2 + \alpha_1} - \zeta_1$ gives $\alpha_1 = \eta_1^2 + 2\eta_1\zeta_1$. Adding $2\eta_1^2 A_1^\intercal P A_1$ to both sides of (19) and substituting $\alpha_1 = \eta_1^2 + 2\eta_1\zeta_1$ gives exactly (18). Thus the condition (10) of Thm. 7 is satisfied by $\eta_1 = \sqrt{\zeta_1^2 + \alpha_1} - \zeta_1$. Applying Thm. 7 completes the proof.

If all robustness bounds η_i in Theorem 7 are chosen proportional to $\sqrt{\zeta_i^2 + \alpha_i} - \zeta_i$ (like in Cor. 10), we obtain the following corollary:

Corollary 11. Suppose the system in (8) is mean-square stable with $\mathbb{E}[\gamma_{ti}] = 0$, $\mathbb{E}[\gamma_{ti}^2] = \alpha_i > 0$. Fix a $Q \succeq I$ and the solution $P \succ 0$ to (9). Let $\eta_i > 0$ be scalars which satisfy (10) and are chosen proportional to $\sqrt{\zeta_i^2 + \alpha_i} - \zeta_i$ where ζ_i are scalars which marginally satisfy

$$\frac{1}{\sqrt{\zeta_i^2 + \alpha_i} - \zeta_i}Q + 2\zeta_i A_i^\intercal P A_i \succeq \left(A^\intercal P A_i + A_i^\intercal PA\right)^+. \tag{20}$$

Then the deterministic system in (11) is stable for any $0 \leq \mu_i < \eta_i$ where the η_i are upper bounded by

$$\eta_i < \sqrt{\zeta_i^2 + \alpha_i} - \zeta_i < \sqrt{\alpha_i}$$

Proof. The proof proceeds by contradiction. Suppose

$$\eta_i = \sqrt{\zeta_i^2 + \alpha_i} - \zeta_i$$

From (20) and using an argument identical to Corollary 10 we have

$$Q + 2\eta_i\zeta_i A_i^\intercal P A_i \succeq \eta_i \left(A^\intercal P A_i + A_i^\intercal PA\right)^+$$

Summing over all the noises,

$$pQ + \sum_{i=1}^p 2\eta_i\zeta_i A_i^\intercal P A_i \succeq \sum_{i=1}^p \eta_i \left(A^\intercal P A_i + A_i^\intercal PA\right)^+ \tag{21}$$

Substituting $\alpha_i = \eta_i^2 + 2\eta_i\zeta_i$, the matrix inequality in (10) reduces to

$$\begin{aligned}
pQ + \sum_{i=1}^p 2\eta_i\zeta_i A_i^\intercal P A_i &\succeq \sum_{i=1}^p \eta_i\left(A^\intercal P A_i + A_i^\intercal PA\right)^+ \\
&+ \sum_{i=1}^p \sum_{j\neq i} \eta_i\eta_j \left(A_i^\intercal P A_j + A_j^\intercal P A_i\right)^+
\end{aligned}$$

which is a contradiction; we need the additional terms

$$\sum_{i=1}^p \sum_{j\neq i} \eta_i\eta_j \left(A_i^\intercal P A_j + A_j^\intercal P A_i\right)^+ \tag{22}$$

on the right-hand side of (21) in order to match (10) in Theorem 7, which shows that the bounds η_i must be less than $\sqrt{\zeta_i^2 + \alpha_i} - \zeta_i$.

Corollaries 10 and 11 go towards showing the functional dependence of upper bounds of the robustness margins on the multiplicative noise variance, namely a $\sqrt{\alpha_i}$ relation. Significantly, the robustness margins collapse to nothing when the variances are all zero and increase monotonically with increasing noise variances.

3.3 Conservative simplifications

It can be shown that $\frac{1}{\sqrt{\zeta_i^2 + \alpha_i} - \zeta_i}$ is convex in ζ_i, so any linearization (first-order Taylor series expansion) will be a global underestimator of this function. Thus a conservative solution can be found by linearization, yielding a convex semidefinite constraint which can be expressed as a generalized eigenvalue problem which can be solved efficiently. For example, linearizing $\frac{1}{\sqrt{\zeta_i^2 + \alpha_i} - \zeta_i}$ about $\zeta_i = 0$ yields $\frac{1}{\sqrt{\alpha_i}} + \frac{1}{\alpha_i}\zeta_i$. This is worked out in the following lemma:

Lemma 12. Define A, A_i, α_i, P, Q as in Cor. 11. Let λ_i be the maximum generalized eigenvalue which solves

$$\left[\left(A^\intercal P A_i + A_i^\intercal PA\right)^+ - \frac{1}{\sqrt{\alpha}}Q\right]v = \lambda_i\left[\frac{1}{\alpha}Q + 2A_i^\intercal P A_i\right]v.$$

Then $\zeta_i \geq \lambda_i$ satisfies (20).

Proof. By Lemma 4 we have the semidefinite bound

$$\lambda_i\left(\frac{1}{\alpha_i}Q + 2A_i^\intercal P A_i\right) \succeq \left(A^\intercal P A_i + A_i^\intercal PA\right)^+ - \frac{1}{\sqrt{\alpha_i}}Q.$$

Rearranging,

$$\left(\frac{1}{\sqrt{\alpha_i}} + \frac{1}{\alpha_i}\lambda_i\right)Q + 2\lambda_i A_i^\intercal P A_i \succeq \left(A^\intercal P A_i + A_i^\intercal PA\right)^+.$$

Since $\frac{1}{\sqrt{\lambda_i^2 + \alpha_i} - \lambda_i}$ is a convex function of λ_i,

$$\frac{1}{\sqrt{\lambda_i^2 + \alpha_i} - \lambda_i} \geq \frac{1}{\sqrt{\alpha_i}} + \frac{1}{\alpha_i}\lambda_i$$

and thus

$$\frac{1}{\sqrt{\lambda_i^2 + \alpha_i} - \lambda_i}Q + 2\lambda_i A_i^\intercal P A_i \succeq \left(A^\intercal P A_i + A_i^\intercal PA\right)^+$$

which is exactly the constraint in (20) with $\zeta_i = \lambda_i$. Noting that $\frac{1}{\sqrt{\zeta_i^2 + \alpha_i} - \zeta_i}$ is nondecreasing in ζ_i completes the proof.

Similarly, an even more conservative bound is obtained by neglecting the contribution of $2\zeta_i A_i^\intercal P A_i$ in (20), again resulting in a generalized eigenvalue problem.

Lemma 13. Define A, A_i, α_i, P, Q as in Cor. 11. Let λ_i be the maximum generalized eigenvalue which solves

$$\left(A^\intercal P A_i + A_i^\intercal PA\right)^+ v = \lambda_i Q v.$$

Then $\zeta_i \geq \frac{1}{2}\left(\alpha\lambda_i - \frac{1}{\lambda_i}\right)$ satisfies (20).

Proof. By Lemma 4 we have the semidefinite bound

$$\lambda_i Q \succeq \left(A^\intercal P A_i + A_i^\intercal PA\right)^+$$

Setting

$$\lambda_i < \frac{1}{\sqrt{\zeta_i^2 + \alpha_i} - \zeta_i}$$

and rearranging yields

$$\zeta_i \geq \frac{1}{2}\left(\alpha\lambda_i - \frac{1}{\lambda_i}\right)$$

and

$$\left(\frac{1}{\sqrt{\zeta_i^2 + \alpha_i} - \zeta_i}\right) Q \succeq (A^\mathsf{T} P A_i + A_i^\mathsf{T} P A)^+$$

Adding $2\zeta_i A_i^\mathsf{T} P A_i \succeq 0$ to the left side gives exactly the constraint in (20).

4. ROBUSTNESS VIA STABILITY OF AUXILIARY SYSTEMS

Now, instead of requiring the same Lyapunov function to ensure mean-square stability of a stochastic system and stability of a perturbed deterministic system with the same nominal A, we construct auxiliary stochastic systems whose mean-square stability implies deterministic stability of the "target" perturbed deterministic system. Such an approach can be fundamentally more flexible than using a shared Lyapunov function since the open-loop dynamics of the auxiliary system are permitted to be significantly less stable.

Theorem 14. Suppose the stochastic system

$$x_{t+1} = \left(A\sqrt{1 + \sum_{i=1}^p \eta_i} + \sum_{i=1}^p \gamma_{ti} A_i\right) x_t$$

with $\mathbb{E}[\gamma_{ti}^2] = \alpha_i \geq \eta_i\left(1 + \sum_{j=1}^p \eta_j\right)$, $\eta_i \geq 0$ is mean-square stable. Then the deterministic system

$$x_{t+1} = \left(A + \sum_{i=1}^p \mu_i A_i\right) x_t$$

is stable for all $|\mu_i| < \eta_i$.

Proof. Mean-square stability implies there exists P such that

$$\begin{aligned}
P &\succ A^\mathsf{T} P A + \sum_{i=1}^p \eta_i A^\mathsf{T} P A + \sum_{i=1}^p \eta_i A_i^\mathsf{T} P A_i \\
&\quad + \sum_{i=1}^p \sum_{j=1}^p \eta_i \eta_j A_i^\mathsf{T} P A_i \\
&\succeq A^\mathsf{T} P A + \sum_{i=1}^p \eta_i (A^\mathsf{T} P A_i + A_i^\mathsf{T} P A) \\
&\quad + \sum_{i=1}^p \sum_{j=1}^p \eta_i \eta_j A_i^\mathsf{T} P A_i \quad (23) \\
&\succeq A^\mathsf{T} P A + \sum_{i=1}^p \eta_i (A^\mathsf{T} P A_i + A_i^\mathsf{T} P A) \\
&\quad + \sum_{i=1}^p \eta_i^2 A_i^\mathsf{T} P A_i + \sum_{i=1}^p \sum_{j\neq i} \eta_i \eta_j A_i^\mathsf{T} P A_j \\
&= \left(A + \sum_{i=1}^p \eta_i A_i\right)^\mathsf{T} P \left(A + \sum_{i=1}^p \eta_i A_i\right)
\end{aligned}$$

By symmetry of the terms $\sum_{i=1}^p \eta_i A^\mathsf{T} P A_i + \sum_{i=1}^p \eta_i A_i^\mathsf{T} P A_i$, the same argument can be applied for each sign combination of η_i i.e. $\pm\eta_1, \pm\eta_2, \ldots, \pm\eta_p$ from (23) onward, which together prove stability of $A + \sum_{i=1}^p k_i \eta_i A_i$ for any $k_i \in \{-1, +1\}$ with the same Lyapunov matrix P. By an argument from Schur complements (see e.g. Corless (1994); Boyd et al. (1994)), this is necessary and sufficient for any

convex combination of $A + \sum_{i=1}^p k_i \eta_i A_i$ to be also stable using P, completing the proof.

Remark 15. The condition $\mathbb{E}[\gamma_{ti}^2] = \alpha_i \geq \eta_i\left(1 + \sum_{j=1}^p \eta_j\right)$ places an upper bound on the robustness margins η_i which is related to the multiplicative noise variances α_i. In the case of $p = 1$, this reduces to $\eta_1 < \frac{1}{2}\left(\sqrt{1 + 4\alpha_1} - 1\right)$.

At first glance the condition of Thm. 14 may seem overly restrictive since it requires mean-square stability with a *scaled* A matrix; indeed such a procedure is somewhat limiting in the open-loop setting since this can make the plant unstable. However, in the control design setting this *essentially does not matter* since the gain can be made larger to compensate, and because a simple scaling of A does not affect controllability of the pair (A, B); to see this, simply note that the rank of the controllability matrix $\begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}$ is unaffected by a nonzero scaling of A. The work of Bernstein (1987) similarly leverages this fact.

5. INPUT UNCERTAINTIES AND ROBUST CONTROL DESIGN ALGORITHMS

In the case where there are uncertainties in the input matrix B under closed-loop state feedback, Theorems 7 and 14 are easily modified by simply substituting

$$\begin{aligned}
A &\leftarrow A + BK, \\
\{A_i\} &\leftarrow \{A_i\} \cup \{B_j K\}, \\
\{\alpha_i\} &\leftarrow \{\alpha_i\} \cup \{\beta_j\}, \\
p &\leftarrow p + q
\end{aligned}$$

yielding a set of $p+q$ robustness bounds $\{\eta_i'\} = \{\eta_i\} \cup \{\psi_j\}$ which ensure stability of

$$x_{t+1} = \left(A + BK + \sum_{i=1}^p \mu_i A_i + \sum_{j=1}^q \nu_j B_j K\right) x_t \quad (24)$$

where $0 \leq \mu_i < \eta_i$, $0 \leq \nu_j < \psi_j$ (bounds in negative directions also assured for Thm. 14). These results are formulated as Algorithms 1 and 2 for generating optimal, maximally robust controllers. Note that Algorithm 1 gives unidirectional bounds while Algorithm 2 gives bidirectional bounds; it is useful to retain the unidirectional bounds of Algorithm 1 in order to realize the potentially larger robustness margins in opposing directions.

6. NUMERICAL RESULTS

Here we consider an inverted pendulum with a torque-producing actuator whose dynamics have been linearized about the vertical equilibrium. In continuous-time the dynamics are

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ m_c & 0 \end{bmatrix}}_{A_c} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B_c} u$$

where m_c is a normalized mass constant. A forward Euler discretization with step size Δt yields

$$A = I + A_c \Delta t = \begin{bmatrix} 1 & \Delta t \\ m_c \Delta t & 1 \end{bmatrix}, \quad B = B_c \Delta t = \begin{bmatrix} 0 \\ \Delta t \end{bmatrix}$$

Uncertainty on the mass constant m_c corresponds to uncertainty on the $(2, 1)$ entry of A. We consider an

Algorithm 1: Robust control design

Input: Controllable nominal pair (A, B), cost matrices $Q \succ 0$, $R \succ 0$, uncertainty directions A_i, B_j and magnitudes $\theta_i > 0$, $\phi_j > 0$.

Output: Gain matrix K and margins η_i, ψ_j such that (24) is stable for all $0 \le \mu_i < \eta_i$, $0 \le \nu_j < \psi_j$.

Define scalar z and scaled multiplicative noise variances $\alpha_i = \theta_i \times z$, and $\beta_j = \phi_j \times z$

Find the largest z^* which still admits a solution to $P = \mathrm{gare}(A, B, Q, R, \alpha_i, \beta_j, A_i, B_j)$ via bisection

Define scalar y and scaled uncertainty magnitudes $\eta_i = \theta_i \times y$, $\psi_j = \phi_j \times y$

Find the largest scaling y^* via bisection which satisfies

$$Q + K^\intercal R K + \sum_{i=1}^{p+q} \alpha_i' A_i'^\intercal P A_i'$$

$$\succeq \sum_{i=1}^{p+q} \eta_i' \left(A_i'^\intercal P (A + BK) + (A + BK)^\intercal P A_i' \right)^+$$

$$+ \sum_{i=1}^{p+q} \sum_{j=1}^{p+q} \eta_i' \eta_j' \left(A_i'^\intercal P A_j' + A_j'^\intercal P A_i' \right)^+$$

where $\{A_i'\} = \{A_i\} \cup \{B_j K\}$, $\{\alpha_i'\} = \{\alpha_i\} \cup \{\beta_j\}$, and $\{\eta_i'\} = \{\eta_i\} \cup \{\psi_j\}$

Return control law

$$K = -\left(R + B^\intercal P B + z^* \sum_{j=1}^{q} \phi_j B_j^\intercal P B_j \right)^{-1} B^\intercal P A$$

and margins $\eta_i = \theta_i \times y^*$, $\psi_j = \phi_j \times y^*$

Algorithm 2: Robust control design

Input: Controllable nominal pair (A, B), cost matrices $Q \succ 0$, $R \succ 0$, uncertainty directions A_i, B_j and magnitudes $\theta_i > 0$, $\phi_j > 0$.

Output: Gain matrix K and robustness margins η_i, ψ_j such that (24) is stable for all $|\mu_i| < \eta_i$, $|\nu_j| < \psi_j$.

Define scalar y and scaled uncertainty magnitudes $\eta_i = \theta_i \times y$, $\psi_j = \phi_j \times y$

Define scaled multiplicative noise variances $\alpha_i = \eta_i \left(1 + \sum_{j=1}^{p} \eta_j + \sum_{k=1}^{q} \psi_k\right)$, and $\beta_j = \psi_i \left(1 + \sum_{i=1}^{p} \eta_i + \sum_{k=1}^{q} \psi_k\right)$

Define scalar $z(y) = \sqrt{1 + \sum_{i=1}^{p} \eta_i + \sum_{j=1}^{q} \psi_i}$, and scaled system matrices $A_z = A \times z$, $B_z = B \times z$

Find the largest y^* which still admits a solution to $P = \mathrm{gare}(A_z, B_z, Q, R, \alpha_i, \beta_j, A_i, B_j)$ via bisection

Return control law

$$K = -\left(R + B_z^\intercal P B_z + \sum_{j=1}^{q} \beta_j B_j^\intercal P B_j \right)^{-1} B_z^\intercal P A_z$$

where quantities P, β_j and z are evaluated at y^*, and margins $\eta_i = \theta_i \times y^*$, $\psi_j = \phi_j \times y^*$

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example where the true mass constant is $\bar{m}_c = 10$, but the nominal model underestimates it as $m_c = 5$; such a situation could easily arise during the initial phase of system identification in adaptive control with noisy measurements, or in time-varying scenarios such as a robot arm picking up a heavy load. We take a step size $\Delta t = 0.1$. The problem data is then

$$\bar{A} = \begin{bmatrix} 1 & 0.1 \\ 1 & 1 \end{bmatrix}, \ A = \begin{bmatrix} 1 & 0.1 \\ 0.5 & 1 \end{bmatrix}, \ \bar{B} = B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},$$

$$Q = R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ \theta_1 = 1$$

Applying Algorithms 1, 2, and certainty-equivalent control design, we obtained the results in Table 6. We found the sets of true $\bar{A}$ matrices stabilized by the controls from Algos. 1 and 2 were $\bar{A} \in \begin{bmatrix} 1 & 0.1 \\ 0.1 + \mu_1 & 1 \end{bmatrix}$ where $|\mu_1| < 6.997$ and $|\mu_1| < 3.970$ respectively. Stability of all systems within these sets was empirically verified by a fine grid search using 10000 samples of μ_1 in each interval. Both robustness sets happened to include the true matrix $\bar{A}$, so the robust controls were guaranteed to stabilize the true system, confirmed by $\rho(\bar{A} + \bar{B}K) < 1$. By contrast, the certainty-equivalent control failed to stabilize the true system. This can be understood intuitively; the pendulum had a larger mass in reality than in the nominal model, so a larger control effort was necessary to stabilize the pendulum and prevent it from falling over. Although on this particular example Algorithm 1 gave a larger (unidirectional) robustness margin, in general this not need hold; certain problem instances admit much larger robustness margins using Algorithm 2 relative to Algorithm 1. Thus, our two algorithms may be considered complementary from a control design standpoint.

Code which implements this example is available at https://github.com/TSummersLab/robust-control-multinoise.

7. CONCLUSION AND FUTURE WORK

This work gives an effective methodology for certifying robustness and designing robust controllers with favorable properties and flexibility relative to competing approaches.

Direct extensions to this work include finding sharper bounds, e.g., via alternate auxiliary systems analogous to the one used in Section 4, and handling nonlinear dependence of the dynamics and/or noise on states and inputs. Future work will integrate the results of this work with adaptive model-based learning control for an end-to-end control framework which gracefully transitions from maximal robustness to maximal performance according to empirical uncertainties.

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Table 1. Stability results for robust control of an inverted pendulum.

Parameter	Open-loop	Certainty-equivalent	Algorithm 1	Algorithm 2
K	[0 0]	[−9.139 −4.153]	[−103.876 −19.851]	[−104.528 −19.948]
$\rho(\bar{A} + \bar{B}K)$	1.316	1.019	0.222	0.225
$\rho(A + BK)$	1.223	0.833	0.060	0.020
η_1	-	-	6.997	3.970
$\max\limits_{0 \le \mu_1 < \eta_1} \rho(A + BK + \mu_1 A_1)$	-	-	0.841	0.632

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