Leveraging Statistical Prior Knowledge in Adaptive Control of Nonlinear Systems

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***Abstract*— Assuming access to prior knowledge about the unknown system parameter θ, given as a probability prior $\varphi(\theta)$, we study how such information can be incorporated into the adaptive control design of nonlinear systems. To this end, we propose a new parameter estimation law, and show that it promotes convergence of parameter estimates to high-probability regions of the parameter space. Our approach is particularly useful when employed as a single universal controller for a large population of systems with similar dynamic structure but different parameter values. In this scenario, our method, on average, leads to a more accurate parameter estimate, which consequently improves transient performance and accelerates state convergence. Numerical studies confirm the effectiveness of the proposed method.**

I. INTRODUCTION

Consider a dynamical system of the form

$$\dot{x} = f(x) + \Delta(x)^\top \theta + B(x)u, \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the control input, $\theta \in \Theta \subset \mathbb{R}^p$ is an unknown but constant parameter vector, and Θ is a convex set. We also account for the possibility of overparameterization, meaning that different parameter vectors in Θ may correspond to the same dynamics through the term $\Delta(x)^\top\theta$. The functions $f : \mathbb{R}^n \to \mathbb{R}^n$, $B : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $\Delta : \mathbb{R}^n \to \mathbb{R}^{p\times n}$ are known and smooth.

Adaptive control. The goal of adaptive control [4], [19], [15], [33], [2], [12], [32] is to design the control input $u(x(t), t)$ such that the system state converges to the desired trajectory $x_d(t)$, irrespective of the value of θ. Adaptive control is a well-established field with a rich theoretical foundation and numerous practical applications in robotics [34], [37], [44], [13], automated driving [40], energy systems [27], [26], and biomedical engineering [29]. An adaptive control algorithm typically consists of a parameter estimator integrated with a feedback mechanism for the controlled system. Traditional Lyapunov-based adaptive controllers ensure the stability of the closed-loop system and guarantee asymptotic convergence of system states to the desired trajectory. However, their transient performance largely depends on the accuracy of the parameter estimates. Accurate parameter estimation not only reduces large deviations in system states but also improves their convergence rate [15], [35]. Nevertheless, parameter estimates are known to converge to their actual values only if (i) the system trajectory satisfies a strong condition known as persistent excitation (PE) [23], [33], which is typically not met, especially in stabilization ($x_d(t) = 0$) and regulation ($x_d(t) =$ constant) problems due to the insufficient richness of the reference signal, and (ii) the dynamical model (1) is not over-parameterized [25], meaning there exists a unique parameter vector θ representing the underlying system dynamics.

Incorporating prior information. In this paper, we study the adaptive control of nonlinear systems of the form (1) with statistical prior information on the parameter vector θ. In particular, we assume that prior knowledge about the unknown parameter θ is available in the form of a multivariate density function $\varphi : \Theta \to \mathbb{R}$, which may be obtained from early experiments or from the physical properties of the system. We design a new parameter estimation law that leverages this statistical information and show that the proposed law promotes convergence of the parameter estimates to high-probability regions of the parameter space. Interestingly, as a special case, we show that the standard Lyapunov-based estimation law implicitly assumes a Gaussian prior on the system parameter θ. The proposed method is particularly useful when the goal is to control not just a single system but a large population of systems that share a similar dynamic structure yet have different parameter values θ that are (approximately) distributed according to $\varphi(\theta)$. In this scenario, our method, on average, yields more accurate parameter estimates, thereby improving transient performance and accelerating the convergence of the system states. The introduced estimation law can be integrated with any Lyapunov function-based control controller and is applicable to systems with matched and unmatched uncertainties.

Related works. A common technique for incorporating prior bounds on parameter vector θ into adaptive control design is through the use of projection operators [33], [19]. In this method, assuming the feasible set Θ is known, the parameter estimate is projected back onto Θ whenever it leaves the set. This mechanism can be particularly useful in the presence of noise, which may otherwise cause the parameter estimates to diverge. Building on recent advances in machine learning and optimization, estimation laws based on natural gradient descent and mirror descent (see, e.g., [24], [8], [7]) have been proposed [21], [25], [45]. These methods exploit non-Euclidean geometries to achieve desirable properties, such as constraining the parameter estimates within a specific region using logarithmic barrier functions without relying on explicit projection operators, designing inertia-based adaptation gains for robot manipulators [21], [45], and promoting sparsity of the parameter estimates [25]. Our method is conceptually related to these techniques in

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that we also employ natural gradient–like estimation laws; however, we construct the underlying geometry using the prior distribution $\varphi(\theta)$.

Paper organization. The remainder of this paper is structured as follows. Section II reviews the necessary background and fundamental definitions in control and probability theory. Section III formally states the problem and the research question studied in this work. Section IV presents the main contribution, namely, the prior-based parameter estimation law and its convergence properties. Numerical results are reported in Section V. Section VI provides the technical proofs of the theorems presented in the paper. Finally, concluding remarks along with directions for future research are given in Section VII.

Notations and definitions. Throughout the paper, we use the following notations. $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative real numbers. Given a real-valued function $V : \mathbb{R}^n \to \mathbb{R}$, we denote the vector of partial derivatives by

$$\frac{\partial V(x)}{\partial x} = \left[\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_s}\right].$$

A continuous function $\zeta : [0, a) \to \mathbb{R}_{\geq 0}$, with $a > 0$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and satisfies $\zeta(0) = 0$. It belongs to class $\mathcal{K}_\infty$ if $a = \infty$ and $\lim_{r\to\infty} \zeta(r) = \infty$. The function $V(x)$ is said to be positive definite if $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. Moreover, $V(x)$ is radially unbounded if $V(x) \to \infty$ as $\|x\| \to \infty$.

II. PRELIMINARIES

In this section, we review fundamental concepts from the theory of dynamical systems, along with some key definitions from statistics and probability theory. We refer the readers to several books for further details [19], [17], [11], [42], [43].

Definition 1 (Stabilizability)**.** System (1) is stabilizable if there exists a control input $u(x, \theta)$ such that the closed-loop system

$$\dot{x} = f(x) + \Delta(x)^\top \theta + B(x)u(x, \theta)$$

is asymptotically stable, i.e., solutions from every initial state converge to zero, for all $\theta \in \Theta$.

Stabilizability is an intrinsic system property that characterizes whether the system admits a stabilizing feedback law. This property is equivalent to the existence of a control Lyapunov function, defined below

Definition 2 (Control Lyapunov functions [3], [36])**.** A positive definite function $V(x, \theta)$, is a control Lyapunov function (CLF) for (1) if it is radially unbounded in x for all $\theta \in \Theta$, and there exists $\zeta(\cdot, \theta) \in \mathcal{K}_\infty$ such that

$$\inf_{u\in R^m} \left[\frac{\partial V(x, \theta)}{\partial x}(f(x) + \Delta(x)^\top \theta + B(x)u)\right] \leq -\zeta(\|x\|, \theta). \tag{2}$$

CLFs can be constructed via physical insights [38], [26], [1], [16], numerical methods [11], [39], [22], or design techniques such as backstepping and feedback linearization [19], [31], [14]. In general, for a dynamical system of the form (1), CLFs depend on the parameters θ. However, if the uncertainty satisfies the *matching condition*

$$\Delta^\top(x)\, \theta \in \text{span}\{B(x)\}, \tag{3}$$

meaning that the uncertainties lie entirely within the span of the control input matrix (i.e., $\Delta^\top(x) = B(x)\,\omega^\top(x)$ for some known dynamics $\omega : \mathbb{R}^n \to \mathbb{R}^{p\times m}$), then it is possible to construct parameter-independent CLFs for (1). In particular, any CLF $V(x)$ designed for the nominal system $\dot{x} = f(x) + B(x)u$ remains valid for the system (1), since the term $\Delta(x)^\top \theta$ can be canceled via the control input.

CLF based control laws. Given an estimation $\hat{\theta}(t)$ of the system parameter θ, we assume that a controller $u(x, \hat{\theta})$ is already in place that satisfies the inquality

$$\frac{\partial V(x, \hat{\theta})}{\partial x}\Big(f(x) + \Delta(x)^\top \hat{\theta} + B(x)u\Big) \leq -\zeta(\|x\|, \theta). \tag{4}$$

We note that one example of such a controller that satisfies the above inequality is the point-wise minimum norm controller [30] defined as

$$u(x, \hat{\theta}) = \underset{u\in\mathbb{R}^m}{\text{argmin}} \left\{u^\top u \;\middle|\; \text{s.t. (4)}\right\} = \begin{cases} -\dfrac{\alpha\beta}{\alpha^\top \alpha}, & b \geq 0, \\ 0, & \text{else}, \end{cases} \tag{5}$$

where $a = \dfrac{\partial V(x, \hat{\theta})}{\partial x} B(x)$ and

$$\beta = \frac{\partial V(x, \hat{\theta})}{\partial x}\Big(f(x) + \Delta(x)^\top \hat{\theta}\Big) + \zeta(\|x\|, \hat{\theta}).$$

Definition 3 (Log-concave distributions)**.** A probability density function $\varphi(\theta)$ is log-concave if

$$\log\big(\varphi(\lambda\theta_1 + (1-\lambda)\theta_2)\big) \geq \lambda \log(\varphi(\theta_1)) + (1-\lambda) \log(\varphi(\theta_2)),$$

for all θ_1, θ_2 and $\lambda \in [0, 1]$. It is strictly log-concave if the above inequality is strict, meaning the $\geq$ is replaced with $>$.

We note that many distributions of practical interest are log-concave. Table I provides a list of several well-known examples. Moreover, the marginals of a log-concave distribution, as well as the convolution and product (when normalized) of two log-concave probability density functions, are also log-concave. A key property of log-concave distributions is their unimodality, which means that the set

$$\{\theta \in \mathbb{R}^p, \varphi(\theta) \geq \eta\} \tag{6}$$

is convex for all $\eta \geq 0$.

Definition 4 (Bregman divergence)**.** Let $\psi : \Omega \to \mathbb{R}$ be a continuously-differentiable, strictly convex function defined on a convex set Ω, The Bregman divergence associated with ψ for points $p, q \in \Omega$ is defined as

$$D_\psi(p, q) = \psi(p) - \psi(q) - \frac{\partial \psi(q)}{\partial q}^\top (p - q).$$

Note that the Bregman divergence is the difference between the value of ψ at point p and the value of the first-order Taylor expansion of ψ around point q evaluated at point p.

Distribution	PDF $\varphi(\theta)$	Initialization $\hat{\theta}(0)$	Update Gain $\left(K(\hat{\theta})^{-1}\right)$
Multivariate Gaussian	$\dfrac{\exp\left(-\frac{1}{2}(\theta-\mu)^\top\Sigma^{-1}(\theta-\mu)\right)}{(2\pi)^{\frac{d}{2}}\lvert\Sigma\rvert^{\frac{1}{2}}}$	μ	Σ
Exponential	$\lambda_i\exp(-\lambda_i\theta_i)$	0	$1/\lambda_i^2$
Logistic	$\dfrac{\exp(-(\theta_i-u_i)/s_i)}{s_i(1+\exp(-(\theta_i-u_i)/s_i))}$	μ_i	$\dfrac{s_i^2}{2}\cdot\dfrac{(1+e^{(\hat{\theta}_i-\mu_i)/s_i})^2}{e^{(\hat{\theta}_i-\mu_i)/s_i}}$
Rayleigh	$\dfrac{\theta_i}{\sigma_i^2}\exp(-\dfrac{\theta_i^2}{2\sigma_i^2})$	σ_i	$\dfrac{\hat{\theta}_i^2\sigma_i^2}{\hat{\theta}_i^2+\sigma_i^2}$
Gamma	$\dfrac{\lambda_i^{\alpha}}{\Gamma(\alpha_i)}\theta_i^{\alpha_i-1}\exp(-\lambda_i\theta_i)$	$\dfrac{\alpha_i-1}{\lambda_i},$	$\dfrac{\hat{\theta}_i^2}{\alpha_i-1}$
Chi	$\dfrac{\theta_i^{k-1}\exp(-\theta_i^2/2)}{2^{k/2-1}\Gamma(k/2)}$	$\sqrt{k_i-1},$	$\dfrac{\hat{\theta}_i^2}{\hat{\theta}_i^2+k_i-1}$
Chi-Squared	$\dfrac{\theta_i^{k/2-1}\exp(-\theta_i/2)}{2^{k_i/2}\Gamma(k_i/2)}$	k_i-2	$\dfrac{2\hat{\theta}_i^2}{k_i-2}$
Gumbel	$\lambda_i\exp\Big(\lambda_i(\mu_i-\theta_i)-\exp(\lambda_i(\mu_i-\theta_i)\Big)$	μ_i	$\dfrac{1}{\lambda^2}\exp(\lambda_i(\hat{\theta}_i-\mu_i))$
Weibull	$\dfrac{k_i}{\lambda_i}\exp(-\dfrac{\theta_i}{k_i})^{k_i}(\dfrac{\theta_i}{\lambda_i})^{k_i-1}$	$\lambda_i(\dfrac{k_i-1}{k_i})^{1/k_i}$	$\dfrac{\hat{\theta}_i^2}{(k_i-1)\Big(1+k_i(\frac{\hat{\theta}_i}{\lambda_i})^{k_i}\Big)}$
Hyperbolic Secant	$\dfrac{1}{2\sigma_i}\mathrm{sech}(\dfrac{\pi}{2\sigma_i}(\theta_i-\mu_i))$	μ_i	$\dfrac{4\sigma_i^2}{\pi^2}\cosh^2(\dfrac{\pi}{2\sigma_i}(\hat{\theta}_i-\mu_i))$
Laplace	$\lambda_i\exp(-\lambda_i\lvert\hat{\theta}_i-\mu_i\rvert)$	μ_i	$1/\lambda_i^2$

TABLE I: Summary of distributions, initializations, and update gains.

III. Promoting Convergence to the Most Probable Parameter

For the system (1), we define the *interpolating set* as

$$\mathcal{A}=\left\{\hat{\theta}\;\middle|\;\Delta(x(t))^\top\hat{\theta}=\Delta(x(t))^\top\theta,\ \forall t\right\}. \tag{7}$$

The set $\mathcal{A}$ contains all parameter vectors that reproduce the true dynamics $\Delta(x(t))^\top\theta$ along the entire trajectory $x(t)$. In many adaptive control and system identification problems, particularly when the uncertainty term $\Delta(x)^\top\theta$ is over-parameterized, multiple parameter vectors can yield identical observed dynamics [25], [28], [18], meaning that $\mathcal{A}$ is generally not a singleton set. Therefore, we introduce a convex optimization problem formulated as

$$\max_{\theta,\,\eta}\quad \eta \qquad \text{s.t.}\quad \begin{cases}\varphi(\theta)\geq\eta,\\ \theta\in\mathcal{A},\\ \eta\geq 0,\end{cases} \tag{P}$$

which seeks for the most likely parameter vector within the set $\mathcal{A}$. Note that here, $\varphi(\theta)$ denotes the prior probability density function over the parameter space. The solution of (P), denoted by $\theta^\star$, represents the parameter vector that is both *consistent with the system dynamics* and *most probable according to the prior knowledge*. Conceptually, $\theta^\star$ serves as the "*best guess*" of the true parameter when empirical observations are combined with prior information.

With this setup, the central research question studied in this paper is as follows:

> *How can we design an update $\dot{\hat{\theta}}(t)$ such that it promotes convergence to $\hat{\theta}^\star$?*

IV. Prior-based Estimation Law

In this section, we answer the above-mentioned research question affirmatively and present the main result of the paper. As discussed in the previous section, our goal is to design an update law for $\hat{\theta}(t)$ that incorporates the prior information $\varphi(\theta)$. To this aim, we seek a mechanism that shapes the parameter update direction according to the geometry implied by the prior distribution.

Given the multivariate probability density function $\varphi:\Theta\to\mathbb{R}$, we define the following positive definite matrix:

$$K_\varphi(\hat{\theta})=\begin{cases}\nabla^2\Big(\log\varphi(\hat{\theta})\Big)^2, & \text{if } \varphi \text{ is log-concave},\\ -\nabla^2\log\varphi(\hat{\theta}), & \text{if } \varphi \text{ is strictly log-concave}.\end{cases}$$

The following theorem establishes an adaptive update law that explicitly incorporates the prior $\varphi(\theta)$ through $K_\varphi(\hat{\theta})$, while preserving stability and convergence properties analogous to standard adaptive schemes.

Theorem 1 (Estimation law with prior information)**.** *Let $\varepsilon>0$ be a scalar, and $\omega:\mathbb{R}\to\mathbb{R}_{>0}$ be a strictly positive and strictly increasing function, with derivative $\omega_\rho(\rho):=$*

$\frac{\partial \omega(\rho)}{\partial \rho}$. *Furthermore, let* $u(t)$ *be any control signal satisfying condition* (4). *Define the parameter estimation law as*

$$\dot{\hat{\theta}} = K(\hat{\theta})^{-1}\, \omega(\rho)\, \Delta(x)^\top \frac{\partial V(x,\hat{\theta})}{\partial x}, \tag{8a}$$

$$\dot{\rho} = -\frac{\omega(\rho)}{\omega_\rho(\rho)\big(V(x,\hat{\theta})+\varepsilon\big)} \frac{\partial V(x,\hat{\theta})}{\partial \hat{\theta}}\, \dot{\hat{\theta}}. \tag{8b}$$

Also, let the parameter estimate be initialized as $\hat{\theta}(0) = \mathrm{Mode}(\varphi(\theta))$. *Then, the closed-loop signals* $x(t)$ *and* $\hat{\theta}(t)$ *remain bounded, and the system state* $x(t)$ *asymptotically converges to zero.*

To preserve the continuity of the discussion, the proof of Theorem 1 is postponed to Section VI, and we proceed to several elaborative remarks on the results given in the above theorem.

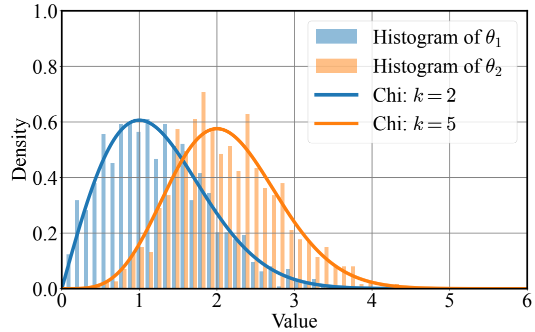
Remark 1 (Interpretation of the update gains)**.** Table I presents the collection of probability density functions along with the corresponding update gains and initial values used in the update law (8a). A general observation is that the update gain tends to be *high* in regions where the probability density function (PDF) has a *low* value, and vice versa. This can be interpreted as follows: when the PDF value is high, it indicates a higher likelihood that the true parameter lies in that region, so the update should proceed more cautiously. Conversely, when the PDF value is low, it suggests a lower likelihood of containing the true parameter, allowing for faster updates. This feature ensures that the parameter trajectory is guided not only by instantaneous data but also by the prior confidence encoded in $\varphi(\theta)$. It is also interesting to note that, according to this table, the constant positive definite gain used in the standard Lyapunov-based design corresponds to a Gaussian prior, where the gain represents the covariance of the prior distribution.

Remark 2. Several modifications can be incorporated into the update laws presented in Theorem 1 to further accelerate convergence or enhance robustness. For example, normalization techniques developed in [9] can improve convergence speed, while σ-modification and dead-zone methods [20], [13] are commonly used to increase robustness.

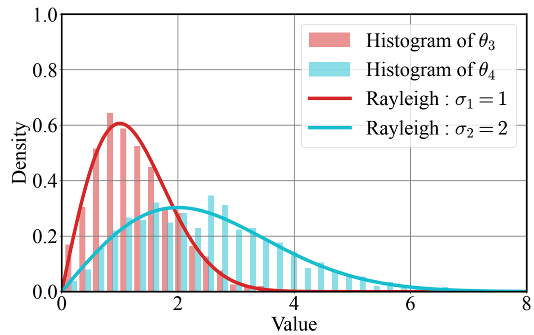
In the following theorem, we establish a connection between the optimization problem (P) and the limiting value of the parameter estimate $\hat{\theta}(t)$. Specifically, we show that, under suitable conditions, the update laws proposed in Theorem 1 guarantee that the parameter estimate $\hat{\theta}(t)$ converges to the optimal solution $\theta^\star$ of optimization problem (P).

Theorem 2. *Consider the update laws proposed in Theorem 1. If* $\lim_{t\to\infty} \hat{\theta}(t) = \hat{\theta}_\infty \in \mathcal{A}$, *then* $\hat{\theta}_\infty = \theta^\star$, *where* $\theta^\star$ *is the solution to the optimization problem* (P).

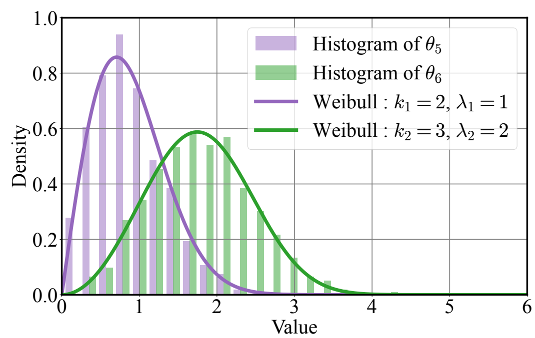
The proof of Theorem 2 is provided in Section VI. This theorem states that if the parameter estimate converges to a point in the set $\mathcal{A}$, then that point must be $\theta^\star$, i.e., the point with the highest probability with respect to $\varphi(\theta)$ within $\mathcal{A}$.



(a) Normalized histograms of θ_1 and θ_2 with fitted Chi distributions.



(b) Normalized histograms of θ_3 and θ_4 with fitted Rayleigh distributions.



(c) Normalized histograms of θ_5 and θ_6 with fitted Weibull distributions.

Fig. 1: Histograms of the system parameter vector $\boldsymbol{\theta}$ for 1000 samples with fitted probability density functions.

V. Numerical Experiments

In this section, we illustrate the performance improvements achieved by using the estimation laws proposed in the previous section. To this aim, Consider the second-order system

$$\dot{x}_1 = x_2, \tag{9}$$

$$\dot{x}_2 = \theta^\top \phi(x) + u, \tag{10}$$

where $x = [x_1, x_2]^\top \in \mathbb{R}^2$ is the system state, u is the control input, $\theta \in \mathbb{R}^8$ is the parameter vector, and $\phi(x)$ is the regressor defined as

$$\phi(x)^\top = \begin{bmatrix} 1 & x_2 & \sin(x_1) & \cos(2x_1) & \sin^2(x_1) & \cos^2(x_1) \end{bmatrix}$$

This system can be interpreted as the dynamics of a single-link manipulator, where the term $\theta^\top \phi(x)$ represents the unknown system dynamics as well as possible unmodeled disturbances and external inputs. In this example, we consider 1,000 instances of the system, each characterized by a different value of the parameter θ. The histograms of these parameter values are shown in Figures 1a, 1b, and 1c. Furthermore, for each entry of the parameter vector θ, a probability distribution $\varphi(\theta_i)$ has been fitted.

By applying Finsler's lemma [41] to (2), we obtain that any quadratic function of the form $V(x) = x^\top P^{-1} x$ serves as a CLF for this system if and only if the positive definite matrix P satisfies the linear matrix inequality

$$B_\perp^\top (AP + PA^\top + \lambda P) B_\perp \leq 0, \tag{11}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B_\perp = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{12}$$

Note that for this system the uncertainty satsfies the matching conditon (3). Hence , the CLf is parmter independent. Figure 2a illustrates the averaged state trajectories of the 1,000 systems, along with the maximum state deviation at each time step. The control law (5) is first applied with the standard update law (i.e., with constant gain) and then with the proposed update law from Theorem 1. It can be observed that the state trajectories exhibit smaller deviations and, on average, converge faster when the proposed update law is used.

VI. Technical proofs

This section presents the proofs of the theorems discussed in the paper.

Proof of Theorem 1. Consider the nonnegative function

$$Q\big(x(t), \hat{\theta}(t), \omega(\rho)\big), = \omega(\rho)\Big(V\big(x(t)\big) + \varepsilon\Big) + D_\psi\big(\theta^\star, \hat{\theta}(t)\big)$$

where

$$\psi(\theta) = \begin{cases} \Big(\log\varphi(\theta)\Big)^2, & \text{if } \varphi \text{ is log-concave,} \\ -\log\varphi(\theta), & \text{if } \varphi \text{ is strictly log-concave.} \end{cases}$$

Taking the time derivative of the above function along the system (1) gives

$$\begin{aligned} \dot{Q} =& \omega(\rho)\frac{\partial V(x,\hat{\theta})}{\partial x}\Big(f(x) + \Delta(x)^\top\theta + B(x)u\Big) \\ &- \omega(\rho)\frac{\partial V(x,\hat{\theta})}{\partial\hat{\theta}}\dot{\theta} + \frac{\omega(\rho)}{\partial\rho}\Big(V\big(x(t)\big) + \varepsilon\Big)\dot{\rho} + \tilde{\theta}^\top K(\hat{\theta})\dot{\hat{\theta}} \\ =& \omega(\rho)\frac{\partial V(x,\hat{\theta})}{\partial x}\Big(f(x) + \Delta(x)^\top\hat{\theta} + B(x)u\Big) \\ &- \omega(\rho)\frac{\partial V(x,\hat{\theta})}{\partial x}\Delta(x)^\top\tilde{\theta} - \omega(\rho)\frac{\partial V(x,\hat{\theta})}{\partial\hat{\theta}}\dot{\theta} \\ &+ \frac{\omega(\rho)}{\partial\rho}\Big(V\big(x(t)\big) + \varepsilon\Big)\dot{\rho} + \tilde{\theta}^\top K(\hat{\theta})\dot{\hat{\theta}} \end{aligned}$$

where $\tilde{\theta} = \hat{\theta} - \theta^\star$. By substituting the control law (4) into the above expression, we have

$$\begin{aligned} \dot{Q} \leq& -\zeta(\|x\|, \hat{\theta}) - \omega(\rho)\frac{\partial V(x,\hat{\theta})}{\partial x}\Delta(x)^\top\tilde{\theta} - \omega(\rho)\frac{\partial V(x,\hat{\theta})}{\partial\hat{\theta}}\dot{\theta} \\ &+ \frac{\omega(\rho)}{\partial\rho}\Big(V\big(x(t)\big) + \varepsilon\Big)\dot{\rho} + \tilde{\theta}^\top K(\hat{\theta})\dot{\hat{\theta}} \end{aligned}$$

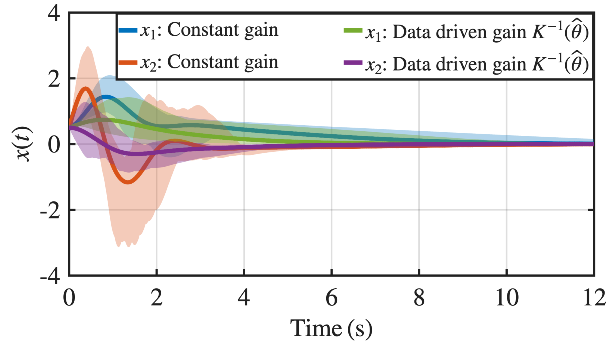
By setting $\dot{\hat{\theta}}$ according to the update law (8a), we find that

$$\dot{Q} \leq -\zeta(\|x\|, \hat{\theta}) + \frac{\omega(\rho)}{\partial\rho}\Big(V\big(x(t)\big) + \varepsilon\Big)\dot{\rho} + \omega(\rho)\frac{\partial V(x,\hat{\theta})}{\partial\hat{\theta}}\dot{\theta} \tag{13}$$

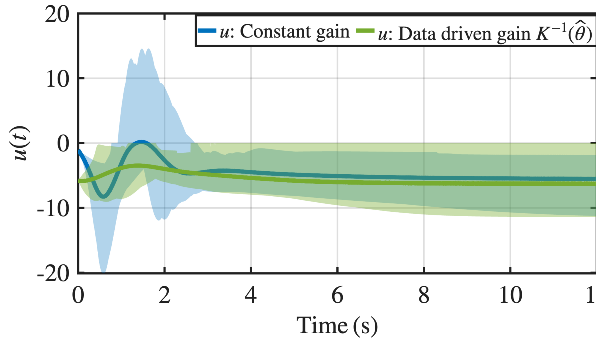
Finally, Substituting update law (8b) into the above inequality yeilds in

$$\dot{Q} \leq -\zeta(\|x\|, \hat{\theta}) \leq 0. \tag{14}$$

As a result, the function $Q\big(x(t), \hat{\theta}(t)\big)$ is non increasing, which implies $x(t)$, $\hat{\theta}(t)$ and $\omega(\rho)$ are bounded. Moreover,



(a) Evolution of state $x(t)$



(b) Evolution of input $u(t)$

Fig. 2: Closed-loop states and control signals of system 10 for the standard (constant-gain) update law and the update law (8a). Results are shown for 1000 systems with different parameter values as given in Fig. 1. The plotted trajectories represent their averages, and the shaded regions indicate the maximum deviations of the states and control signals over time.

we have

$$\begin{aligned}\dot{\zeta}(\|x\|,\hat{\theta}) &= \frac{\partial\zeta(\|x\|,\hat{\theta})}{\partial x}\dot{x} + \frac{\partial\zeta(\|x\|,\hat{\theta})}{\partial\hat{\theta}}\dot{\hat{\theta}} \\ &= \frac{\partial\zeta(\|x\|,\hat{\theta})}{\partial x}\Big(f(x) + \Delta(x)^\top\theta + B(x)u\Big) \\ &\quad + \frac{\partial\zeta(\|x\|,\hat{\theta})}{\partial\hat{\theta}}K(\hat{\theta})^{-1}\omega(\rho)\Delta(x)^\top\frac{\partial V(x,\hat{\theta})}{\partial x}\end{aligned}$$

Since all the signals in the above expression are bound, we conclude that $\dot{\zeta}(\|x\|,\hat{\theta})$ is bound, which implies that $\zeta(\|x\|,\hat{\theta})$ is uniformly continuous. Moreover, taking the time integral from both sides of inequality (14) yields

$$\begin{aligned}\int_0^\infty \zeta(\|x\|,\hat{\theta})dt &\leq Q(x(0),\hat{\theta}(0)) - Q(x(\infty),\hat{\theta}(\infty)) \\ &\leq Q(x(0),\hat{\theta}(0)) < \infty\end{aligned}$$

which implies that the function $\zeta(\|x\|,\hat{\theta})$ is both uniformly continuous and integrable. Hence, by Barbalat's lemma [33], [19], we have $\zeta(\|x\|,\hat{\theta}) \to 0$ as $t \to \infty$. As a result, $x(t)$ asymptotically converges to zero. ■

Proof of Theorem 2. This proof builds on a recent technique for implicit regularization in optimization and adaptive control [6], [5], [25], extending it to adaptive control systems with unmatched uncertainty. . Let θ' be any constant vector of parameters. By taking the time derivative of the Bregman divergence $D_\psi(\theta',\hat{\theta})$, we find that

$$\begin{aligned}\frac{d}{dt}D_\psi(\boldsymbol{\theta},\hat{\boldsymbol{a}}) &= -\left(\frac{d}{dt}\frac{\partial\psi(\hat{\theta})}{\partial\hat{\theta}}\right)(\theta'-\hat{\theta}) = \dot{\hat{\theta}}^\top K(\hat{\theta})(\hat{\theta}-\theta') \\ &=\omega(\rho)\frac{\partial V(x,\hat{\theta})}{\partial x}\Delta(x)^\top\big(\hat{\theta}(t)-\theta'\big)\end{aligned}$$

where the last equality is obtained by setting $\dot{\hat{\theta}}$ according to (8a). Integrating both sides of the above shows that

$$\begin{aligned}D_\psi(\theta',\hat{\theta}(0)) =& D_\psi(\boldsymbol{\theta},\hat{\theta}(t)) \\ &+ \int_0^t \omega(\rho(t))\frac{\partial V(x,\hat{\theta})}{\partial x}\Delta(x)^\top\big(\hat{\theta}(t)-\theta'\big)\,d\tau.\end{aligned} \tag{15}$$

If we now take $\theta' \in \mathcal{A}$, then we have

$$\Delta(x)^\top\theta' = \Delta(x)^\top\theta,$$

which implies that the integral term on the right hand side of (15) becomes independent of θ'. Assuming that $\hat{\theta}(t) \to \hat{\theta}_\infty \in \mathcal{A}$, we can take the limit as $t \to \infty$ and say that for any $\boldsymbol{\theta} \in \mathcal{A}$, $\hat{\boldsymbol{a}}_\infty \in \mathcal{A}$,

$$\begin{aligned}D_\psi(\theta',&\hat{\theta}(0)) = D_\psi(\theta',\hat{\theta}(t)) \\ &+ \int_0^t \omega(\rho(t))\frac{\partial V(x,\hat{\theta})}{\partial x}\big(\Delta(x)^\top\hat{\theta}(t) - \Delta(x)^\top\theta\big)\,d\tau.\end{aligned}$$

Because the only dependence of the right-hand side on θ' is in the first term, and because this relation holds for any θ', the $\arg\min$ of the two Bregman divergences must be identical. The minimum of the right-hand side over θ' is clearly obtained at $\hat{\theta}_\infty$, while the minimum of the left-hand side is by definition obtained at $\arg\min_{\boldsymbol{\theta}\in\mathcal{A}} d_\psi(\boldsymbol{\theta},\hat{\theta}(0))$. From this, we conclude that

$$\hat{\theta}_\infty = \arg\min_{\theta'\in\mathcal{A}} d_\psi(\theta',\hat{\theta}(0)),$$

Note that as mentioned in Theorem 1, we have $\hat{\theta}(0) = \text{Mode}(\varphi(\theta))$. Moreover since $\psi(\theta) = -\log(\varphi(\theta))$ or $\psi(\theta) = (\log(\varphi(\theta)))^2$, we have

$$\begin{aligned}\hat{\theta}_\infty &= \arg\min_{\theta'\in\mathcal{A}} d_\psi(\theta',\hat{\theta}(0)) \\ &= \arg\min_{\theta'\in\mathcal{A}} \psi(\theta') = \underbrace{\arg\max_{\theta'\in\mathcal{A}}}_{(P)} \varphi(\theta') = \theta^\star\end{aligned}$$

which completes the proof. ■

VII. Conclusions And Future Directions

In this paper, we studied how to improve the performance of adaptive systems by leveraging prior knowledge in the form of the distribution of unknown system parameters. We proposed new parameter estimation laws that utilize this prior information by intelligently initializing the estimates and adaptively adjusting the update gains, encouraging convergence toward high-probability regions of the parameter space. Future work may include applying this approach to adaptive compensation for actuator faults in systems with actuator failures [10], as well as extending it to adaptive predictor and observer design [25], [19].

References

[1] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada. Control barrier function based quadratic programs for safety critical systems. *IEEE Transactions on Automatic Control*, 62(8):3861–3876, 2017.

[2] A. M. Annaswamy and A. L. Fradkov. A historical perspective of adaptive control and learning. *Annual Reviews in Control*, 52:18–41, 2021.

[3] Z. Artstein. Stabilization with relaxed controls. *Nonlinear Analysis-Theory Methods & Applications*, 7(11):1163–1173, 1983.

[4] K. J. Åström and B. Wittenmark. *Adaptive Control*. Dover Publications, Mineola, NY, 2nd edition, 2008.

[5] N. Azizan and B. Hassibi. Stochastic gradient/mirror descent: Minimax optimality and implicit regularization. In *International Conference on Learning Representations (ICLR)*, 2019.

[6] N. Azizan, S. Lale, and B. Hassibi. Stochastic mirror descent on overparameterized nonlinear models. *IEEE Transactions on Neural Networks and Learning Systems*, 33(12):7717–7727, 2022.

[7] F. Bach. *Learning theory from first principles*. Adaptive Computation and Machine Learning. The MIT Press, Cambridge, MA, 2024.

[8] A. Beck and M. Teboulle. Mirror descent and nonlinear projected subgradient methods for convex optimization. *Operations Research Letters*, 31(3):167–175, 2003.

[9] M. Boveiri, M. Khosravi, and P. M. Esfahani. Accelerating adaptive systems via normalized parameter estimation laws. *arXiv preprint*, arXiv:2510.17371, 2025.

[10] M. Boveiri and M. S. Tavazoei. Adaptive actuator failure compensation on the basis of contraction metrics. *IEEE Control Systems Letters*, 6:1376–1381, 2022.

[11] S. Boyd and L. Vandenberghe. *Linear Matrix Inequalities in System and Control Theory*. SIAM, 1994.

[12] S. Dean, H. Mania, N. Matni, B. Recht, and S. Tu. Regret bounds for robust adaptive control of the linear quadratic regulator. In S. Bengio, H. Wallach, H. Larochelle, K. Grauman, N. Cesa-Bianchi, and R. Garnett, editors, *Advances in Neural Information Processing Systems*, volume 31. Curran Associates, Inc., 2018.

[13] Z. T. Dydek, A. M. Annaswamy, and E. Lavretsky. Adaptive control of quadrotor uavs: A design trade study with flight evaluations. *IEEE Transactions on Control Systems Technology*, 21(4):1400–1406, 2013.

[14] R. Freeman and P. V. Kokotovic. *Robust Nonlinear Control Design: State-Space and Lyapunov Techniques*. Birkhäuser, 2008.

[15] P. A. Ioannou and J. Sun. *Robust Adaptive Control*. Prentice Hall, 1996.

[16] I. Karafyllis, D. Theodosis, and M. Papageorgiou. Lyapunov-based two-dimensional cruise control of autonomous vehicles on lane-free roads. *Automatica*, 145:110517, 2022.

[17] H. K. Khalil. *Nonlinear systems*. Prentice-Hall, Englewood Cliffs, NJ, USA, 2002.

[18] M. Khosravi and R. S. Smith. The existence and uniqueness of solutions for kernel-based system identification. *Automatica*, 148:110728, 2023.

[19] M. Krstic, I. Kanellakopoulos, and P. Kokotovic. *Nonlinear and Adaptive Control Design*. John Willey, 1995.

[20] M. Krstić and P. V. Kokotović. Modular approach to adaptive nonlinear stabilization. *Automatica*, 31(1):95–100, 1995.

[21] T. Lee, J. Kwon, and F. C. Park. A natural adaptive control law for robot manipulators. In *Proceedings of the IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*, pages 8628–8635, 2018.

[22] I. R. Manchester and J.-J. E. Slotine. Control contraction metrics: Convex and intrinsic criteria for nonlinear feedback design. *IEEE Transactions on Automatic Control*, 62(6):3046–3053, 2017.

[23] K. S. Narendra and A. M. Annaswamy. *Stable Adaptive Systems*. Dover Publications, 2012.

[24] A. Nemirovski, A. Juditsky, G. Lan, and A. Shapiro. Robust stochastic approximation approach to stochastic programming. *SIAM Journal on Optimization*, 19(4):1574–1609, 2009.

[25] J.-J. E. S. Nicholas M. Boffi. Implicit regularization and momentum algorithms in nonlinearly parameterized adaptive control and prediction. *Neural Computation*, 33:590–673, 2021.

[26] R. Ortega, A. Loría, P. J. n. Nicklasso, and H. Sira-Ramírez. *Passivity-based Control of Euler-Lagrange Systems: Mechanical, Electrical and Electromechanical Applications*. Springer, 1998.

[27] D. A. Pierre. A perspective on adaptive control of power systems. *IEEE Transactions on Power Systems*, 2(2):387–395, 1987.

[28] G. Pillonetto, F. Dinuzzo, T. Chen, G. De Nicolao, and L. Ljung. Kernel methods in system identification, machine learning and function estimation: A survey. *Automatica*, 50(3):657–682, 2014.

[29] M. Polycarpou and J. Conway. Indirect adaptive nonlinear control of drug delivery systems. *IEEE Transactions on Automatic Control*, 43(6):849–856, 1998.

[30] J. A. Primbs, V. Nevistic, and J. C. Doyle. A receding horizon generalization of pointwise min-norm controllers. *IEEE Transactions on Automatic Control*, 45(5):898–909, 2000.

[31] R. Sepulchre, M. Jankovic, and P. Kokotovic. *Constructive Nonlinear Control*. Springer, Berlin, Heidelberg, 1997.

[32] G. Shi, K. Azizzadenesheli, M. O' Connell, S.-J. Chung, and Y. Yue. Meta-adaptive nonlinear control: Theory and algorithms. In *Advances in Neural Information Processing Systems*, volume 34, pages 10013–10025. Curran Associates, Inc., 2021.

[33] J.-J. Slotine and W. Li. *Applied Nonlinear Control*. Prentice Hall, Englewood Cliffs, NJ, 1991.

[34] J.-J. E. Slotine and W. Li. On the adaptive control of robot manipulators. *International Journal of Robotics Research*, 6(3):49–63, 1987.

[35] Y. Song, K. Zhao, and M. Krstic. Adaptive control with exponential regulation in the absence of persistent excitation. *IEEE Transactions on Automatic Control*, 62(5):2589–2596, 2017.

[36] E. D. Sontag. A 'universal' construction of artstein's theorem on nonlinear stabilization. *Systems & Control Letters*, 13(2):117–123, 1989.

[37] M. W. Spong. An historical perspective on the control of robotic manipulators. *Annual Review of Control, Robotics, and Autonomous Systems*, 5(Volume 5, 2022):1–31, 2022.

[38] M. W. Spong, S. Hutchinson, and M. Vidyasagar. *Robot Modeling and Control*. Wiley, 2nd edition, 2020.

[39] W. Tan and A. Packard. Stability region analysis using polynomial and composite polynomial lyapunov functions and sum-of-squares programming. *IEEE Transactions on Automatic Control*, 53(2):565–571, 2008.

[40] A. J. Taylor and A. D. Ames. Adaptive safety with control barrier functions. In *2020 American Control Conference (ACC)*, pages 1399–1405, 2020.

[41] F. Uhlig. A recurring theorem about pairs of quadratic forms and extensions: a survey. *Linear Algebra and its Applications*, 25:219–237, 1979.

[42] R. Vershynin. *High-Dimensional Probability: An Introduction with Applications in Data Science*. Cambridge University Press, 2018.

[43] M. J. Wainwright. *High-Dimensional Statistics: A Non-Asymptotic Viewpoint*. Cambridge University Press, 2019.

[44] H. Wang. Adaptive control of robot manipulators with uncertain kinematics and dynamics. *IEEE Transactions on Automatic Control*, 61(2):491–496, 2016.

[45] P. M. Wensing and J.-J. E. Slotine. Beyond convexity – contraction and global convergence of gradient descent, 2018.