Structured Numerical Integration and its Application to High-Dimensional Stochastic Programming

I. PROBLEM STATEMENT AND MOTIVATION

Consider the following optimization problem

$$\min_{\lambda \in \Lambda} \left\{ \int_{\mathbb{R}^n} f(\lambda, x) \mathbb{P}(\mathrm{d}x) \right\}, \tag{1}$$

where $\Lambda \subset \mathbb{R}^d$ is convex and closed, f is real-valued, convex with respect to the first and measurable with respect to the second variable and $\mathbb{P}$ is a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, where $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel sigma algebra of $\mathbb{R}^d$. Problem (1) is the formulation of many *stochastic programming problems*, which is a rich class of problems with applications in many areas of science and engineering ranging from telecommunication and medicine to finance [1]. Most solution methods for (1) require an approximation of $\mathbb{P}$ by a probability measure based on a finite (possibly random or randomized) sample $X_1, \ldots, X_n$ with probabilities $p_1, \ldots, p_n$ and on solving the convex program

$$\min_{\lambda \in \Lambda} \left\{ \sum_{i=1}^{n} p_i f(\lambda, X_i) \right\}. \tag{2}$$

In a nutshell, the task of stochastic programming reduces to the question of how to choose the samples $\{X_i\}_{i=1}^n$ and corresponding weights $\{p_i\}_{i=1}^n$ such that

- Problem (2) is a good approximation of Problem (1), where the approximation error can be quantified;
- Problem (2) can be solved efficiently.

II. GOALS OF THE PROJECT

The aim of this project is to investigate potential methods to efficiently compute approximations of the type (2) to Problem (1). As a first method we would like to study the use of the enormous progress in *Quasi-Monte Carlo* theory and practice, in particular, of randomly shifted lattice rules and to provide theoretical arguments of their superiority over standard Monte Carlo methods [2–4]. A second method that we would like to investigate is importance sampling and the use of recent developments in concentration inequalities [5, 6], where the goal is to derive finite sample guarantees for the approximation (2) with respect to (1), as well as controlling the variance of the mentioned approximation.

Finally, the project should provide some numerical experiments comparing the two studied methods with state-of-the-art approaches. One particularly interesting problem, that naturally falls into the class (1) is the maximum entropy estimation problem subject to moment constraints [7, 8].

III. REQUIREMENTS

The project is well suited for a student that enjoys mathematics. A solid background in analysis and convex optimization is required.

IV. SUPERVISORS

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